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UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

FIBER-OPTIC COMMUNICATIONS

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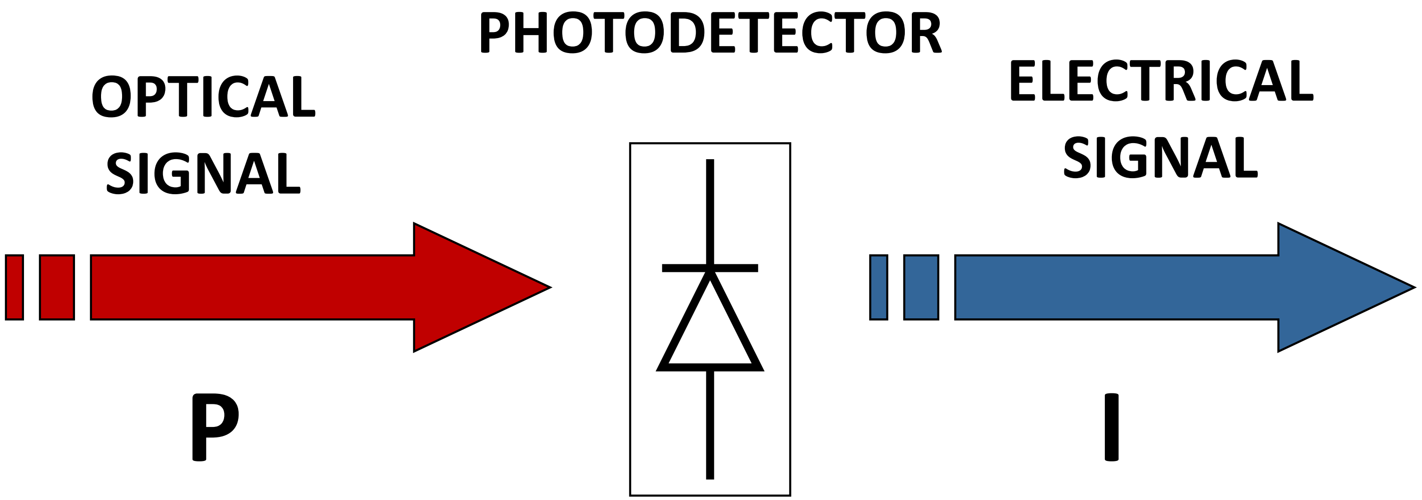
PHOTODIODES

PHOTODIODES

Introduction



QUADRATIC CONVERSION

$$I \propto P$$

General Aspects

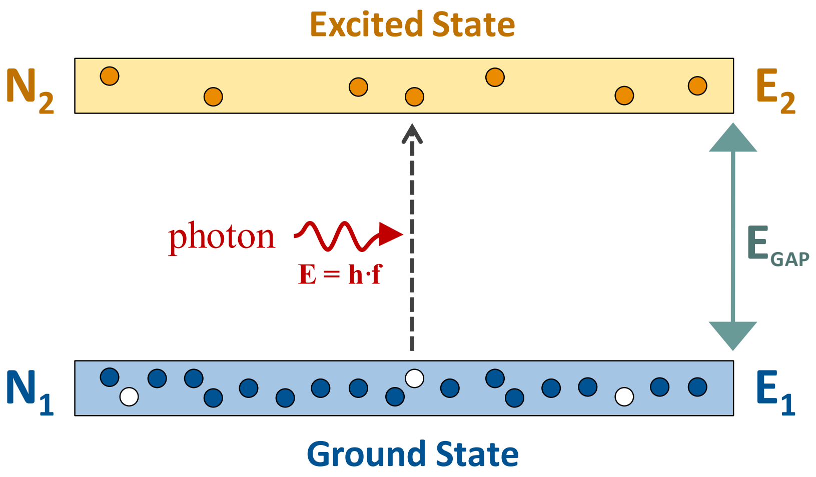
Desirable Characteristics

- ✓ RIGTH FREQUENCY WINDOW
- ✓ HIGH CONVERSION EFFICIENCY
- ✓ LINEAR O/E TRANSFER FUNCTION
- ✓ LARGE O/E BANDWIDTH
- ✓ REDUCED NOISE LEVEL
- ✓ TEMPERATURE STABILITY
- ✓ FIBER SIZE COMPATIBILITY
- ✓ LOW CONSUMPTION
- ✓ REDUCED COST
- ✓ LONG LIFETIME

General Aspects

Working Principle

(STIMULATED) ABSORPTION

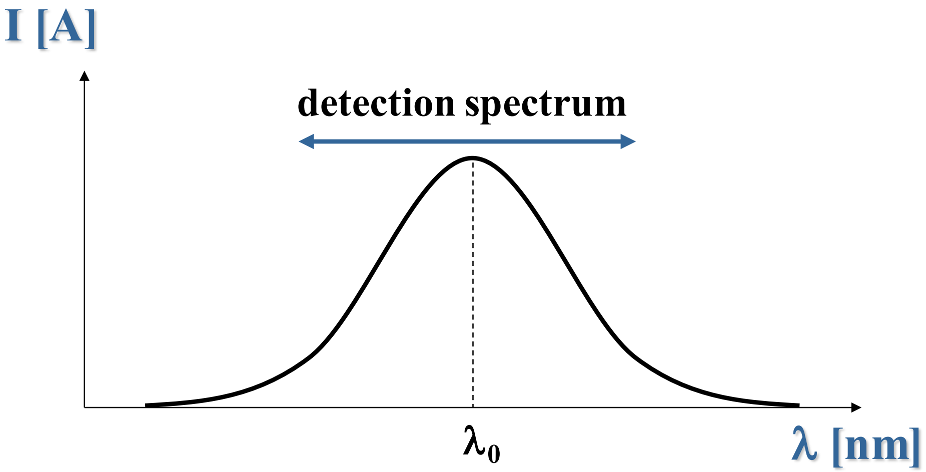


"The incident photon is absorbed by an electron which increments its energy level"

$$hf \geq E_g \rightarrow \lambda \leq \frac{h \cdot c}{E_g} \equiv \lambda_c$$

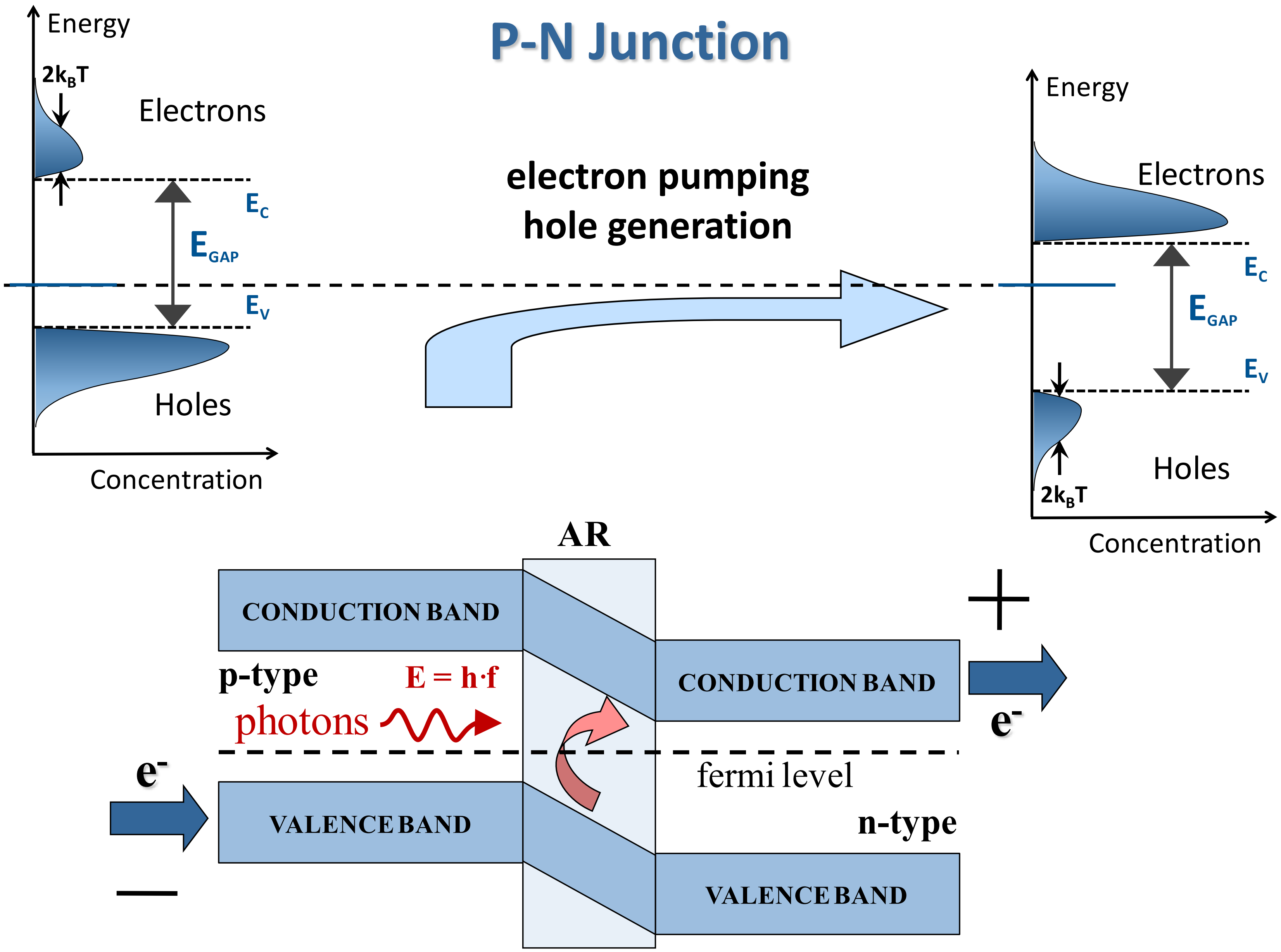
$$\lambda_c = \frac{1.24}{E_g[eV]}[\mu m]$$

cut-off wavelength



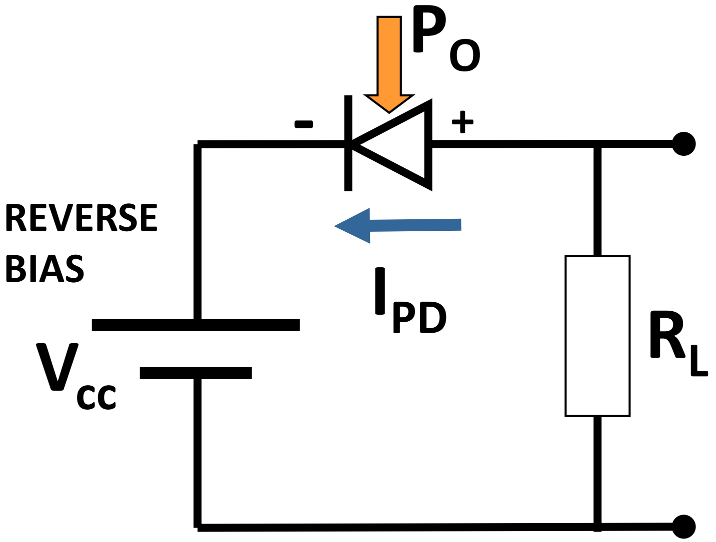
General Aspects

P-N Junction



General Aspects

Circuital Model

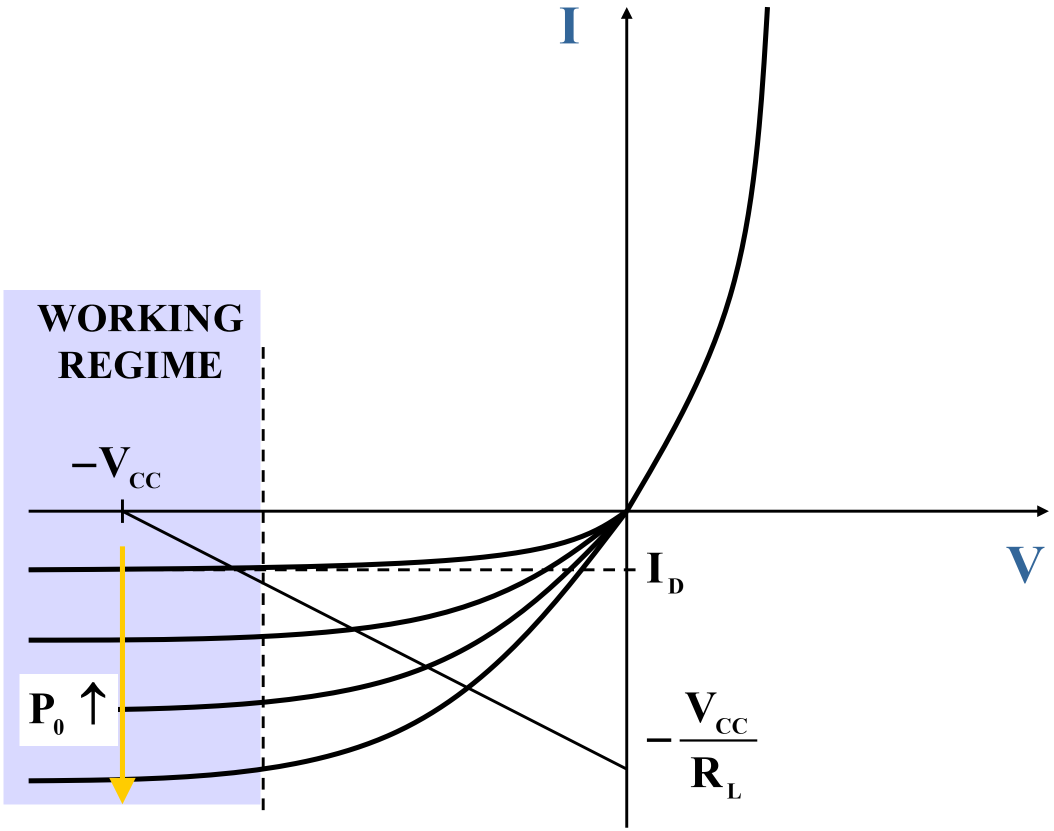


$$I_{PD} = I_R\left(e^{V/V_{Th}} - 1\right)$$

Reverse Saturation Current $I_R = I_D + RP_o$

Thermal Voltage $V_{Th} = kT/q$

I-V CHARACTERISTIC



I_D: dark current

General Aspects

Quantum Efficiency

“Measure of the photon-electron conversion efficiency”

$$\eta \equiv \frac{\langle N^{o}e-h/seg \rangle}{\langle N^{o}fot/seg \rangle} = \frac{I_P/q}{P_{IN}/hf} \leq 1$$

Depends on:

- **materials**
- **structure**

Responsivity

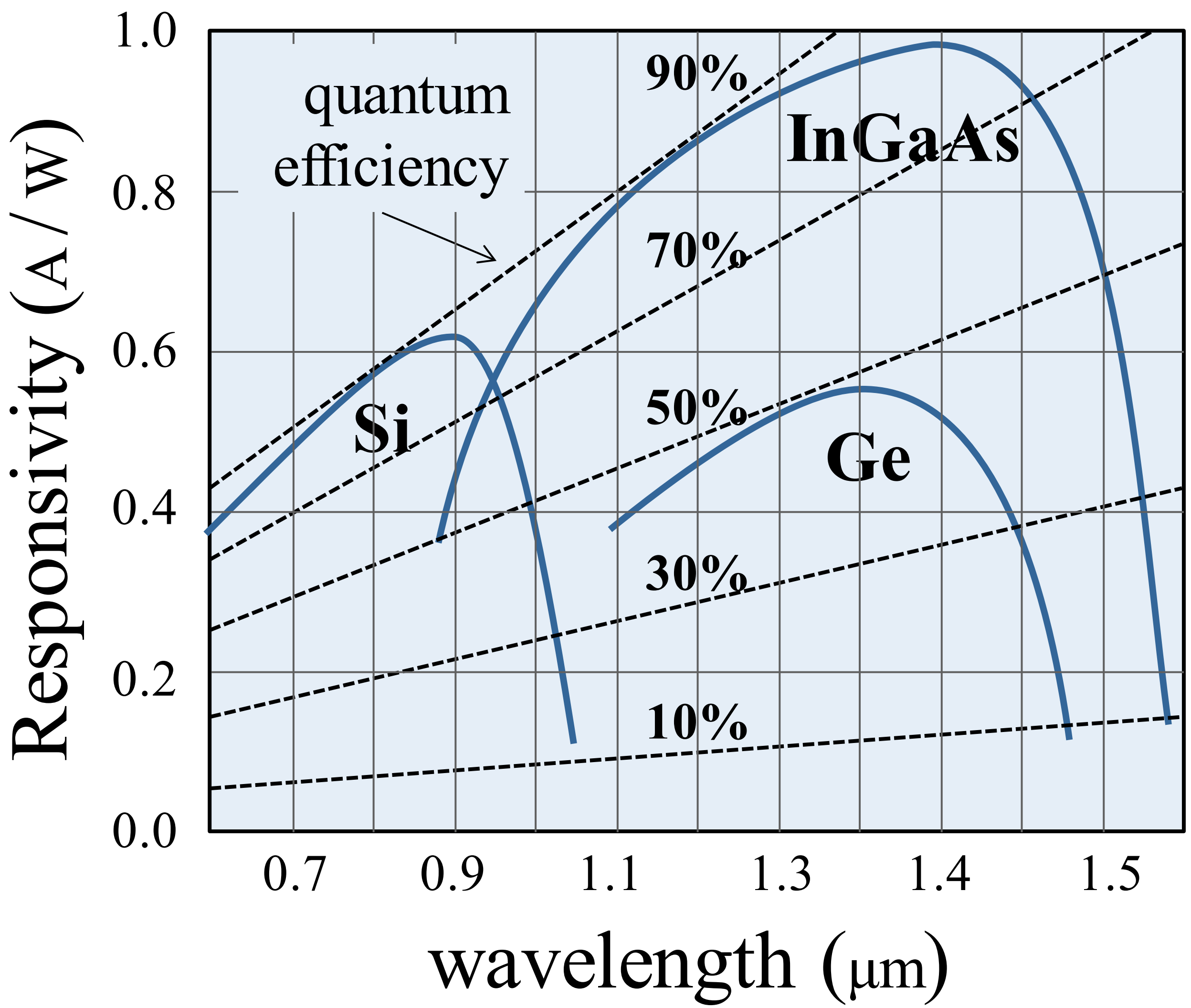
“Average delivered photocurrent over average incident optical power ratio (transfer function)”

$$\Re \equiv \frac{I_P}{P_{IN}} = \eta \frac{q}{hf} = \eta \frac{q}{h} \frac{\lambda}{c} \quad [A/W]$$

$$\lambda \uparrow \rightarrow \Re \uparrow$$

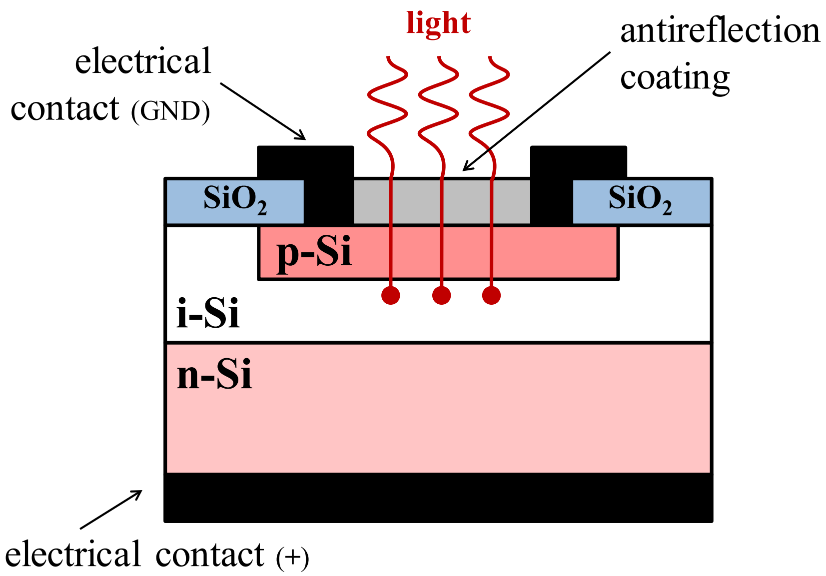
General Aspects

Responsivity for different photodetector materials



Types of Photodiodes

PIN Photodiode

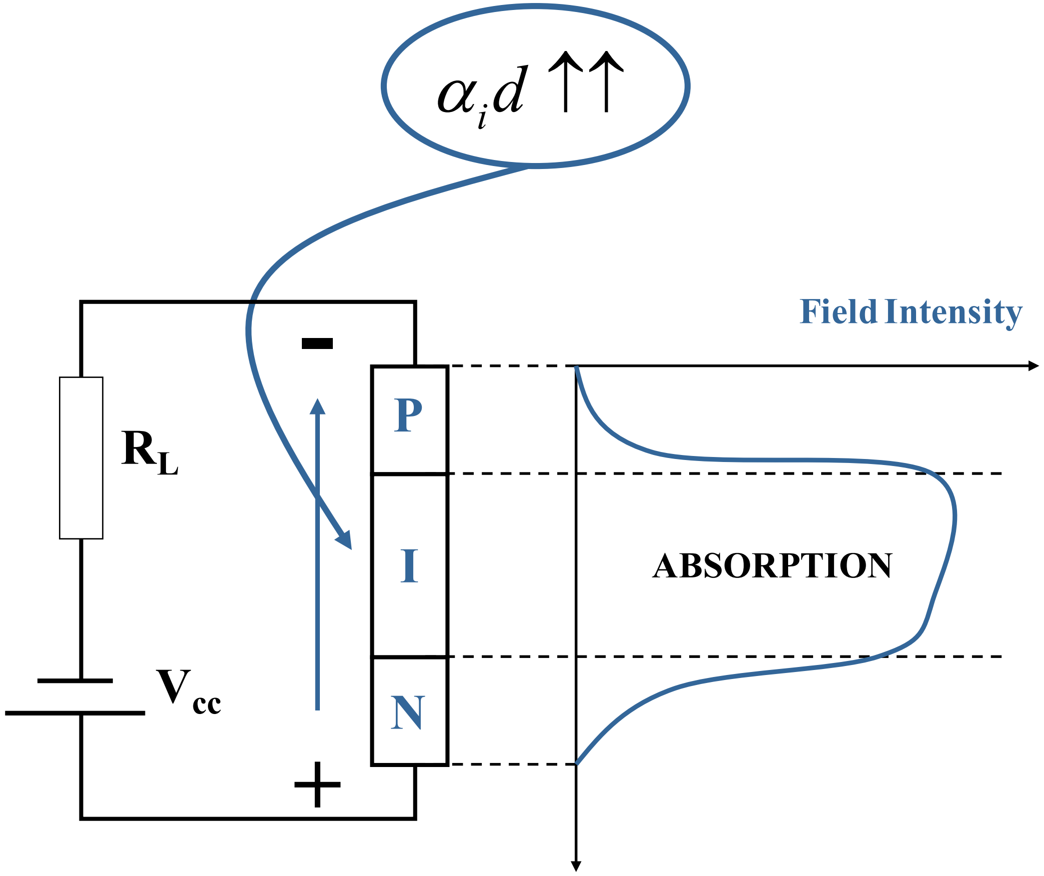


$$\eta \propto 1 - e^{-\alpha_i d} \quad \begin{cases} d \approx 50-100\,\mu m \\ \eta > 90\% \end{cases}$$

trade-off $\quad d\uparrow \begin{cases} \eta \uparrow \\ t_r \uparrow \end{cases}$

t_r: response time

intrinsic semiconductor material

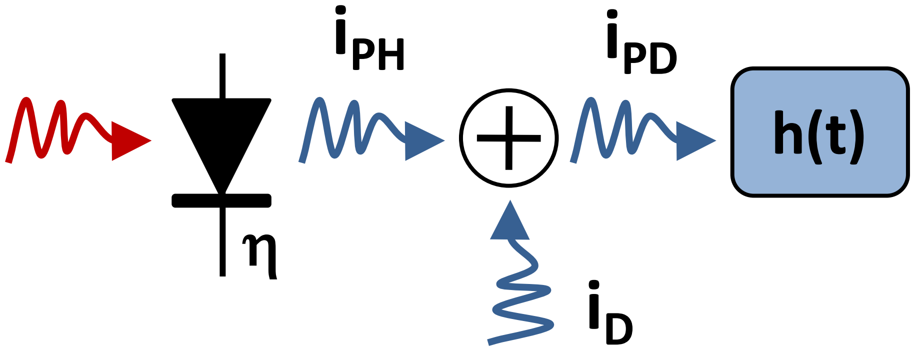


Types of Photodiodes

I –V Characteristic

current = photocurrent + dark current

h(t): impulse response



$$i_{PD} = i_{PH} + i_D$$

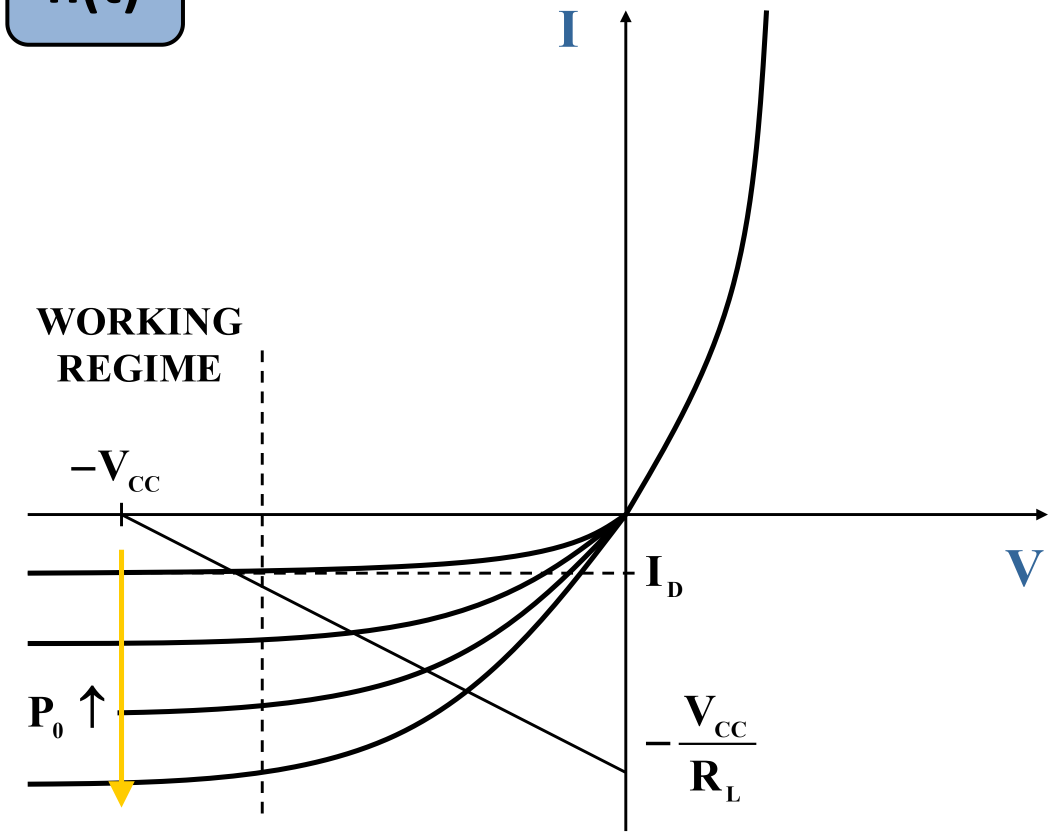
$$I_{PH} \equiv \langle i_{PH} \rangle = \Re \cdot P_{IN}$$

$$\sigma^2_{PH} = 2qB \cdot I_{PH}$$

$$I_D \equiv \langle i_D \rangle$$

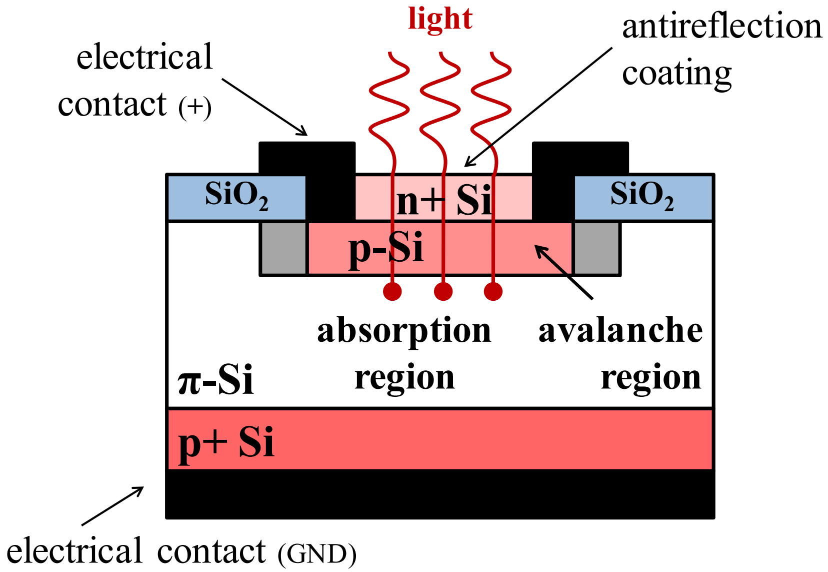
$$\sigma^2_D = 2qB \cdot I_D$$

B: receiver bandwidth



Types of Photodiodes

APD Photodiode AVALANCHE PHOTODIODE

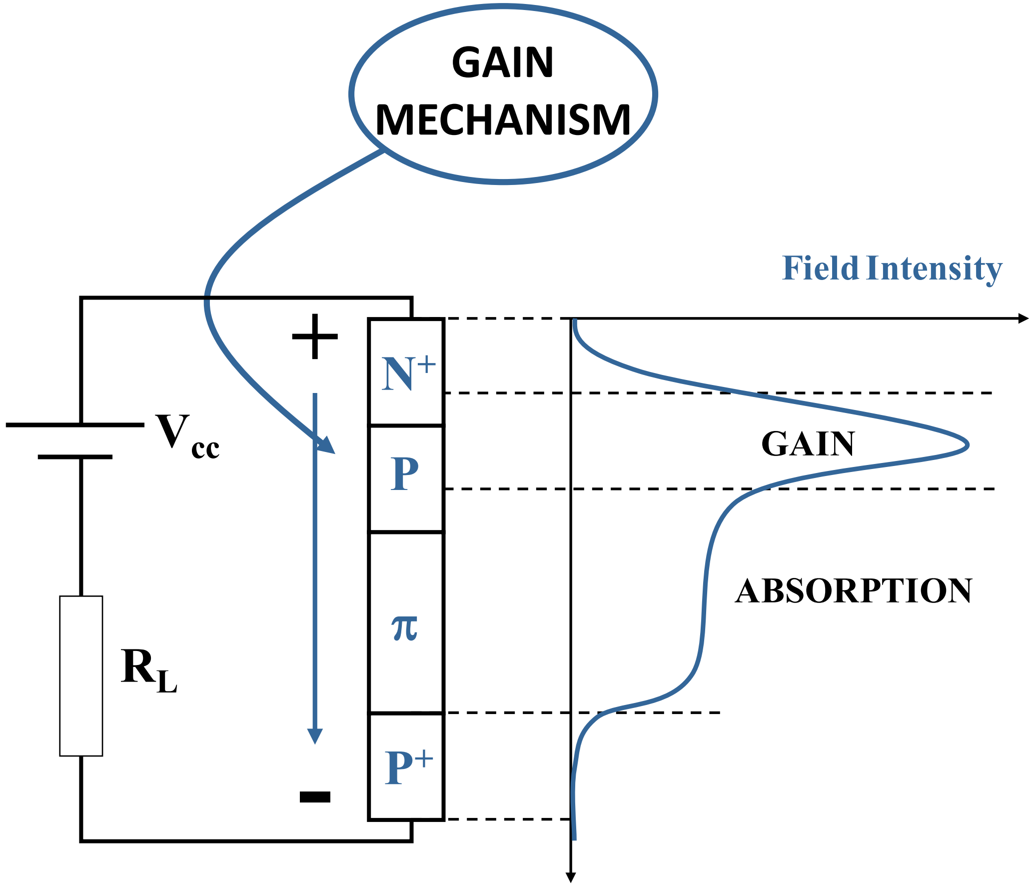


trade-off

$$\Re_{APD} = M \cdot \Re_{PIN}$$

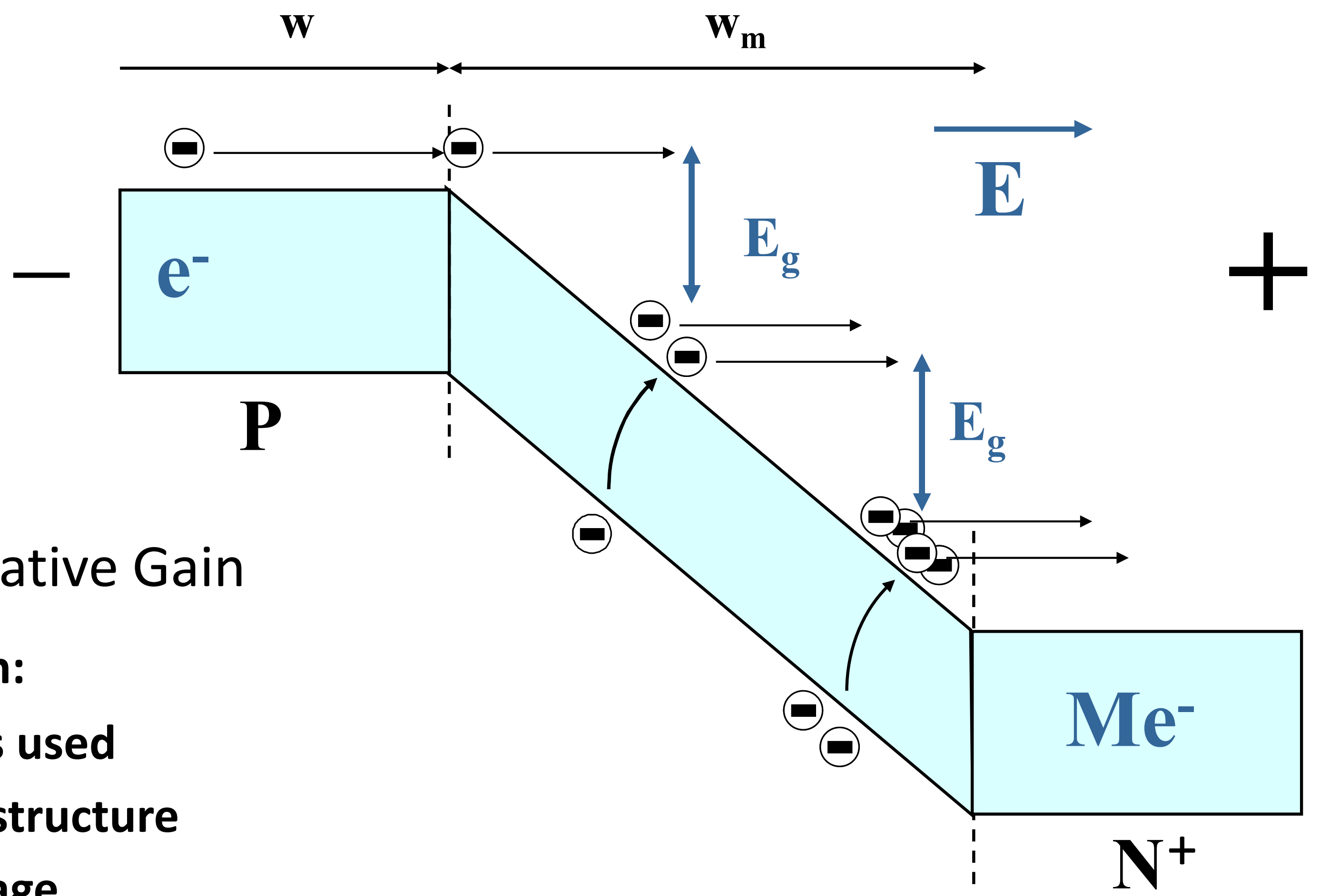
$$\Re_{APD} \approx 20 - 80 \quad A/W$$

$$t_r\big|_{APD} > t_r\big|_{PIN}$$



Types of Photodiodes

Avalanche Effect



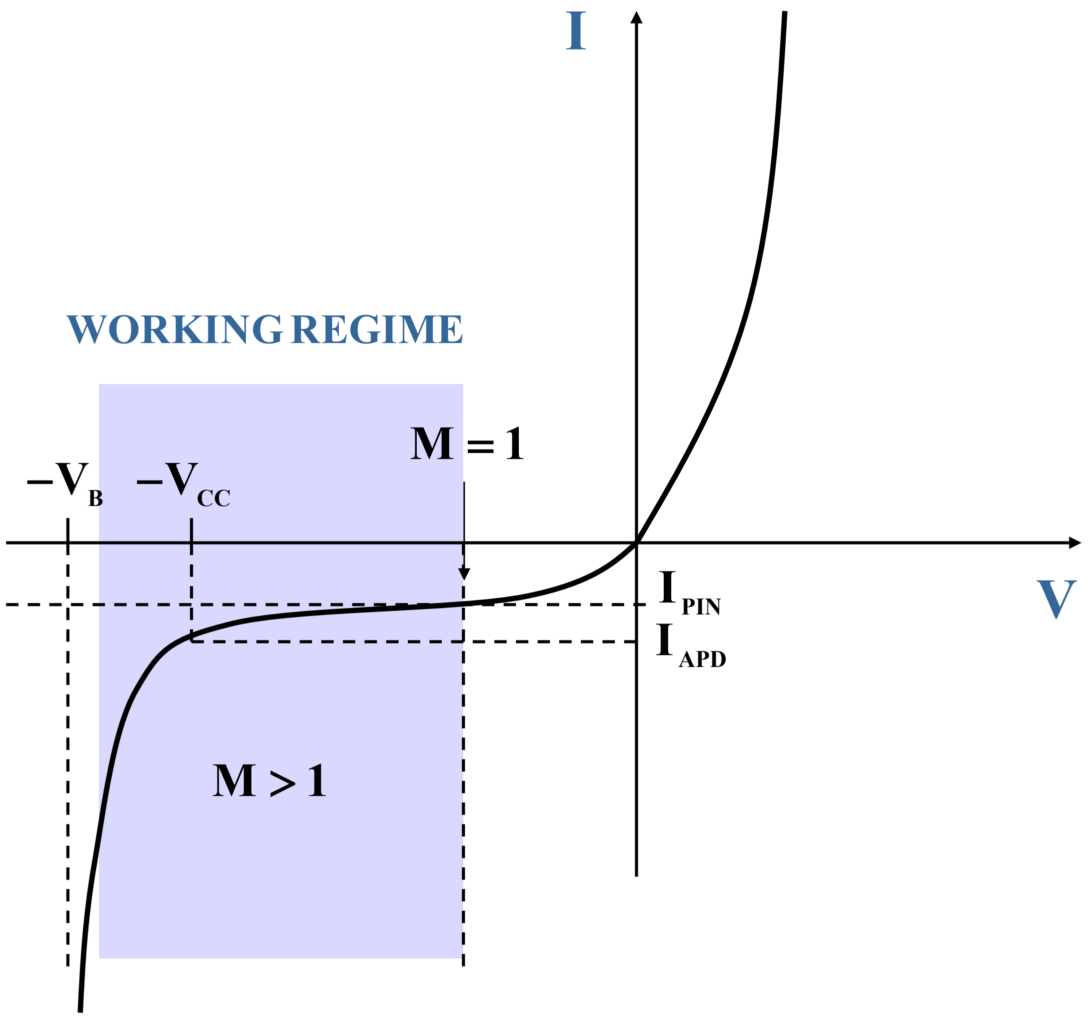
M: Multiplicative Gain

Depends on:

- **materials used**
- **physical structure**
- **bias voltage**
- **temperature**

Types of Photodiodes

I –V Characteristic



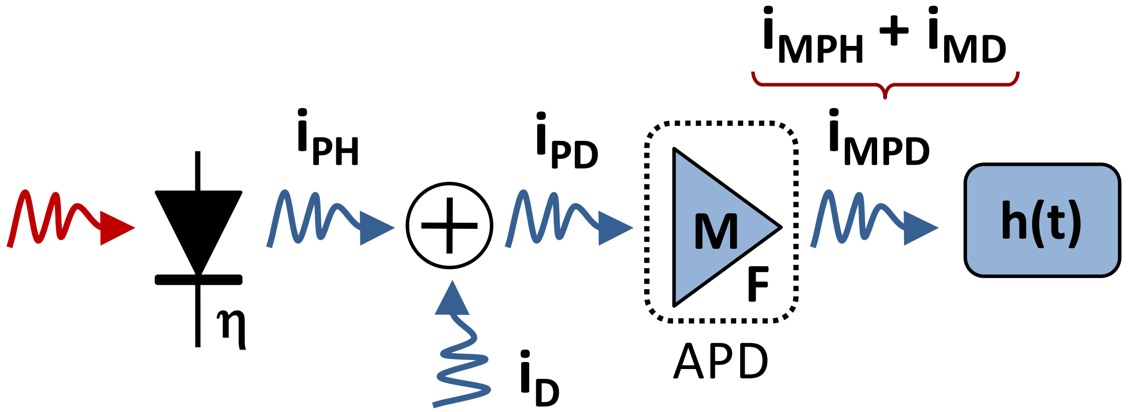
Gain Parameter

$$M = \frac{1}{1-\left(V_{cc}/V_B\right)^n}$$

V_B: breakdown bias

Types of Photodiodes

Equivalent Statistical Model of an APD



i_{PD}: primary current
i_{MPD}: secondary current
i_{PH}: photocurrent
i_{MPH}: amplified photocurrent
i_D: dark current
i_{MD}: amplified dark current
M: gain F: noise factor
h(t): impulse response

$$i_{MPH}\begin{cases}\langle i_{MPH}\rangle \equiv I_{MPH} = M\cdot I_{PH}\\ \sigma^2_{MPH} \approx M^2F(M)\cdot \underbrace{\sigma^2_{PH}}_{2qB\cdot I_{PH}}\end{cases}$$

$$i_{MD}\begin{cases}\langle i_{MD}\rangle \equiv I_{MD} = M\cdot I_{D}\\ \sigma^2_{MD} \approx M^2F(M)\cdot \underbrace{\sigma^2_{D}}_{2qB\cdot I_{D}}\end{cases}$$

$$I_{PH} = \Re\cdot P_{IN}$$

$$I_{MPD} = M\cdot\underbrace{\left(I_{PH}+I_D\right)}_{I_{PD}} = M\left(\Re\cdot P_{IN}+I_D\right)$$

$$\sigma^2_{MPD} \approx M^2F(M)\cdot\underbrace{\left(\sigma^2_{PH}+\sigma^2_D\right)}_{\sigma^2_{PD}} = M^2F(M)\cdot 2qB\left(\Re\cdot P_{IN}+I_D\right)$$

Types of Photodiodes

Noise Factor (F)

Approx. expression ⟶ $F(M) \approx M^x \qquad 0'2 < x < 1$

	x
Si	0.3
InGaAs	0.7
Ge	1

Empirical expression ⟶ $F(M) = kM + (1-k)\left(2 - 1/M\right) \qquad 0 < k < 1$

$$k = 0 \rightarrow F \approx 2 \quad \text{(ideal)}$$
$$k = 1 \rightarrow F = M$$
$$M = 1 \quad \text{(PIN)} \quad \rightarrow F = 1$$

	k
Si	0.015 - 0.035
InGaAs	0.3 - 0.5
Ge	0.6 - 1

Responsivity – Bandwidth trade-off

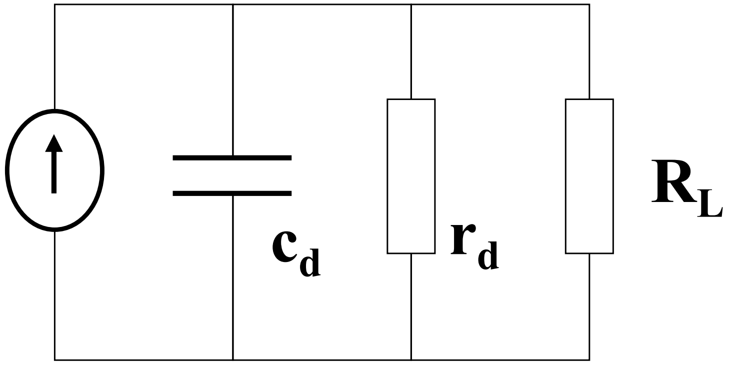
1. AR propagation
2. ⇨ Temporal constant $R_L \cdot c_d$
3. ⇨ Avalanche Effect

PIN → BW=ct

APD → M·BW=ct

Types of Photodiodes

Equivalent Circuit

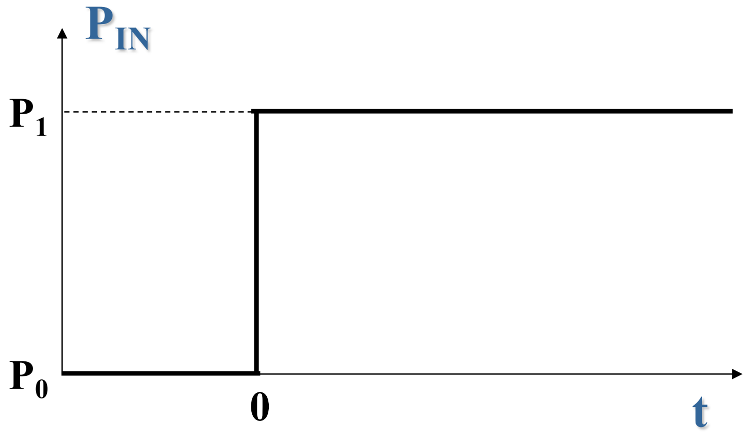


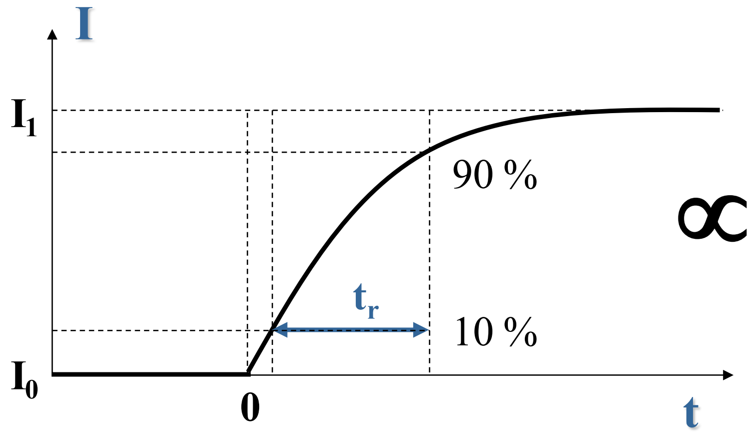
$r_d >> R_L$

Response Time

$$t_r = \ln\left(\frac{0.9}{0.1}\right)\cdot \underbrace{R_L c_d}_{\tau} \approx 2'19\cdot R_L c_d$$

$$B_{3dB} = \frac{1}{2\pi\cdot R_L c_d}$$





$\propto e^{\tau\cdot t}$

InGaAs – PIN → B_{3dB} = 40-50 GHz

InGaAs – APD → B_{3dB} < 1 GHz

Types of Photodiodes

APD vs PIN

ADVANTAGES

- Better Sensitivity (5-15 dB)
- Reduction of P_{IN} fluctuations

DRAWBACKS

- Lower Bandwidth
- Noise Enhancement
- Temperature Control
- Higher Cost
- Higher Consumption

PHOTODETECTION NOISE

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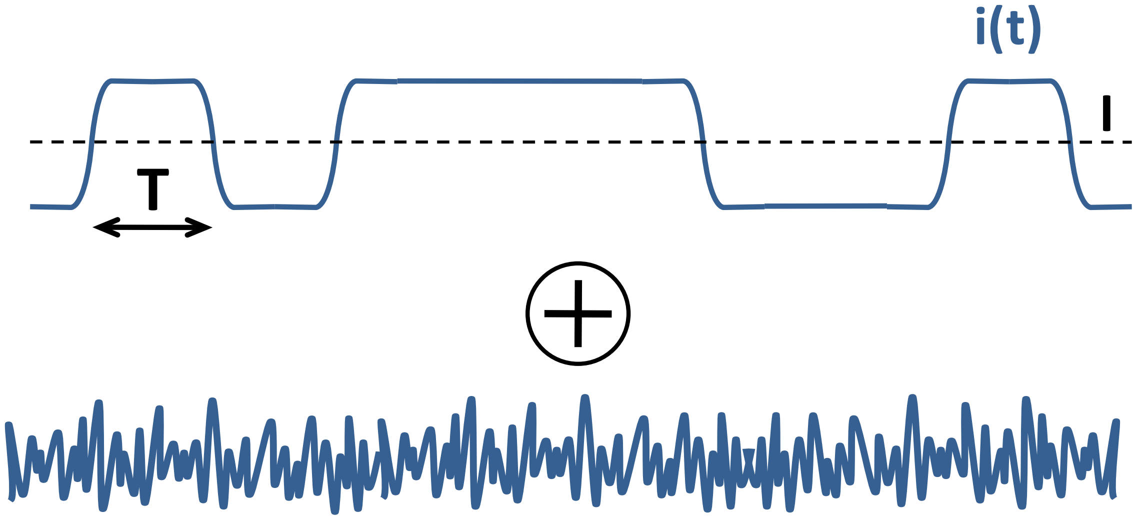
PHOTODETECTION NOISE

Definition

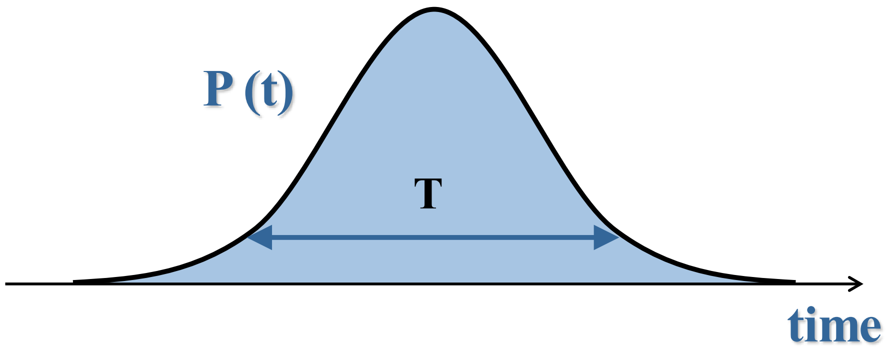
"Random perturbation of the transmitted signal which can mask the information contained in it to the point of making the detection impossible"

"Random fluctuations of the electrical current delivered by the photodetector circuit"



Light´s Random Nature

Deterministic Concepts



Power → $\mathbf{P(t)}$

Bit Energy → $\mathbf{E_{bit} = \int P(t)dt}$

Photon Energy → $\mathbf{hf}$

Random Concepts

$$\text{N}^{\text{o}}\ \text{fot/bit} \equiv \frac{E_{bit}}{hf} = m = \underbrace{\langle m \rangle}_{\text{INFO}} + \underbrace{\left(m - \langle m \rangle\right)}_{\text{FLUCTUATION}}$$

m: random variable

Light´s Random Nature

SNR CONCEPT

$$N^o\ \text{fot/bit} \equiv m = \underbrace{\langle m \rangle}_{\text{INFO}} + \underbrace{\left(m - \langle m \rangle\right)}_{\text{FLUCTUATION}}$$

$$SNR \equiv \frac{\langle m \rangle^2}{\left\langle \left(m - \langle m \rangle\right)^2 \right\rangle} = \frac{\langle m \rangle^2}{\sigma_m^2} < \infty$$

Light Randomness

LASER

coherent light → Poisson statistics

$$\sigma_m^2 = \langle m \rangle$$

$$SNR = \langle m \rangle$$

LED

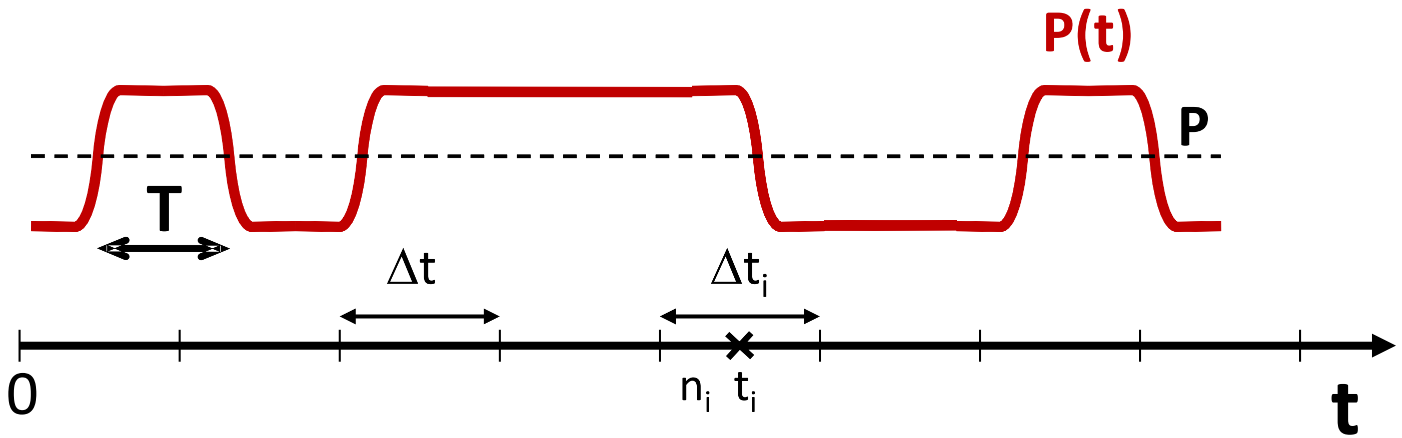
incoherent light → Bose-Einstein statistics

$$\sigma_m^2 = \langle m \rangle \left(\langle m \rangle + 1 \right)$$

$$SNR = \frac{\langle m \rangle^2}{\langle m \rangle \left(\langle m \rangle + 1 \right)} \approx 1$$

Light´s Random Nature

Photon arrival statistics → Poisson (coherent light)



1. The number of photons arrived in a given temporal window is independent of the number of photons arrived in any other non-overlapping and disjoint temporal window.

2. Number of received photons in the ith interval:

$$n_i \begin{cases} 1 & p_i = \lambda_n(i \cdot \Delta t)\Delta t \\ 0 & 1 - p_i \end{cases}$$

$$\lambda_n(t) = \frac{P(t)}{hf}$$

Average number of photons per unit time

Light´s Random Nature

3. Average number of photons arrived in a given temporal window:

$$\langle n_T \rangle (t) = \int_t^{t+T} \lambda_n(\tau)\, \partial\tau = \underbrace{\int_t^{t+T} \frac{P(\tau)}{hf}\, \partial\tau}_{\text{PHOTONS IN } (t,t+T)}$$

Average number of photons received in (t,t+T)

4. Probability of receiving k photons during a given temporal window:

$$p_T(k,t) \equiv \frac{\langle n_T \rangle^k (t)}{k!} e^{-\langle n_T \rangle (t)}$$

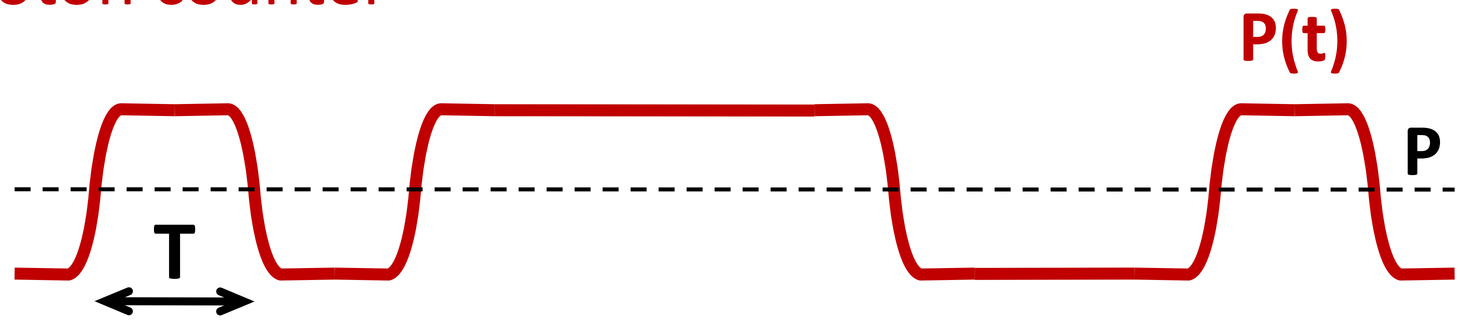
Probability of receiving k photons during the interval (t,t+T)

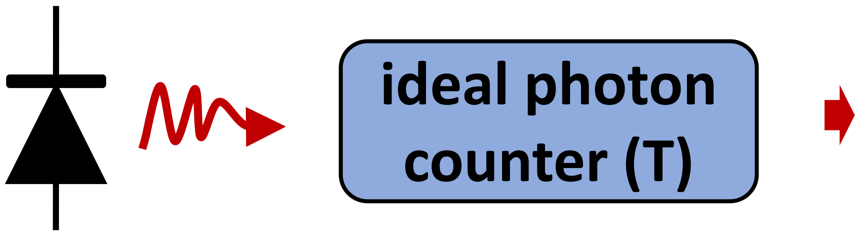
5. The variance of the number of received photons equals its mean value:

$$\sigma^2_{n,T}(t) = \langle n_T \rangle (t)$$

Photodetection Noise

Ideal photon counter





$n_T(t)$

Poisson
Random
Variable

mean

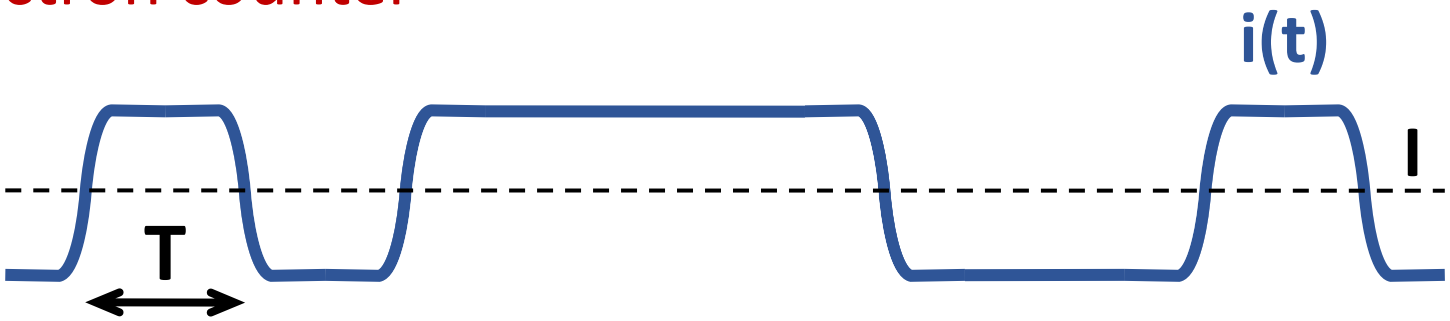
$$\langle n_T \rangle(t) = \frac{1}{hf} \int_{t}^{t+T} P(\tau)\, \partial\tau$$

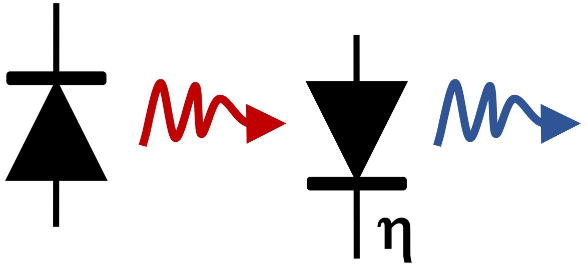
variance

$$\sigma^2_{n,T}(t) = \langle n_T \rangle(t)$$

Photodetection Noise

Ideal electron counter





ideal electron counter (T) ➔ $m_T(t)$ Poisson Random Variable

mean

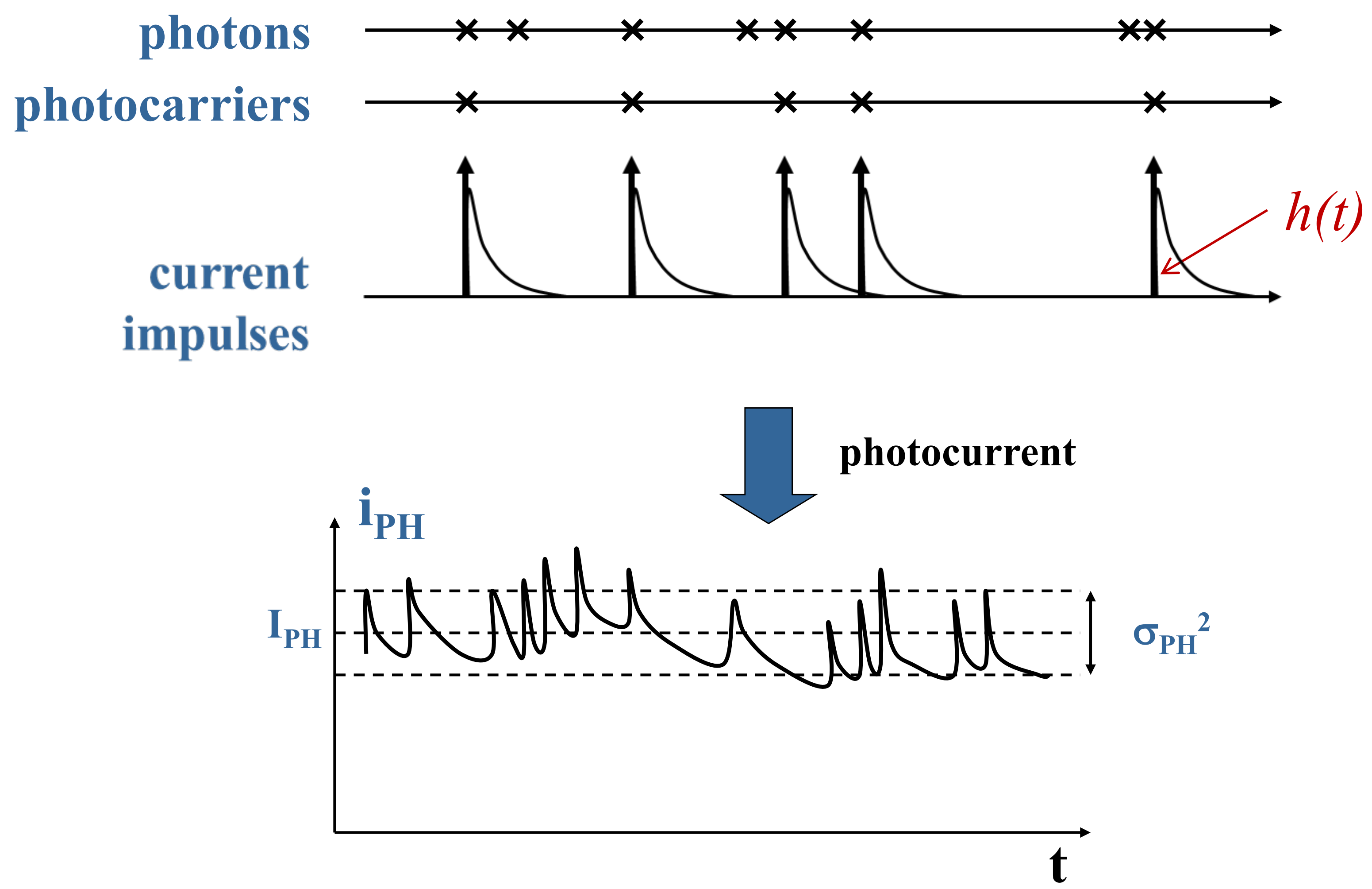
$$\langle m_T\rangle(t) = \eta\langle n_T\rangle(t) = \frac{\eta}{hf}\int_{t}^{t+T} P(\tau)\,\partial\tau$$

variance

$$\sigma^2_{m,T}(t) = \langle m_T\rangle(t)$$

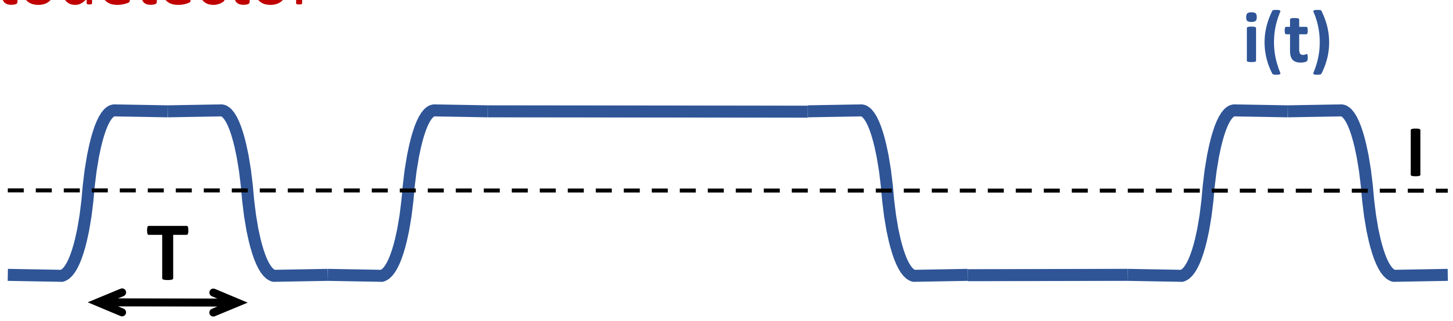
Photodetection Noise

Shot Noise

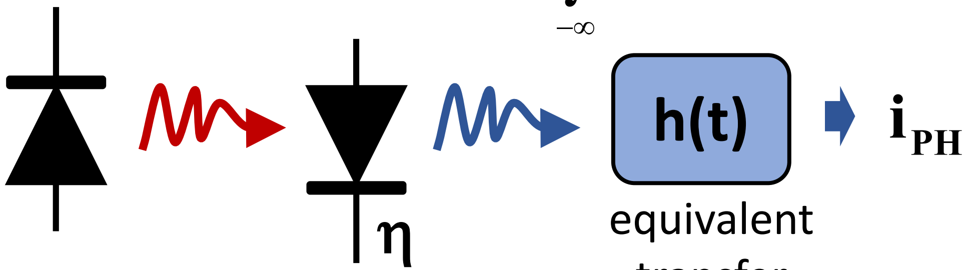


Photodetection Noise

PIN photodetector



$$\int_{-\infty}^{\infty} h(\tau)\,\partial\tau = q$$



equivalent transfer function

Instantaneous photocurrent

mean

$$\langle i_{PH}\rangle(t) = \frac{\eta}{hf}\int_{0}^{\infty} P(\tau)\,h(t-\tau)\,\partial\tau$$

variance

$$\sigma^2_{PH}(t) = \frac{\eta}{hf}\int_{0}^{\infty} P(\tau)\,h^2(t-\tau)\,\partial\tau$$

Proof ➡ Appendix

Photodetection Noise

Mean

slow signal approximation

$$h(t) << P(t)$$

$$\frac{\eta}{hf}\int_0^\infty P(\tau)h(t-\tau)\partial\tau \approx \eta\frac{P(t)}{hf}\underbrace{\int_0^\infty h(t-\tau)\partial\tau}_{q} = \underbrace{\eta\frac{q}{hf}}_{\Re}P(t) = \boxed{\Re P(t) = \langle i_{PH}\rangle(t)}$$

Constant Power

$$P = ct \longrightarrow \begin{array}{l} I_{PH} = \Re P \\ \sigma_{PH}^2 = 2qB\,I_{PH} \end{array}$$

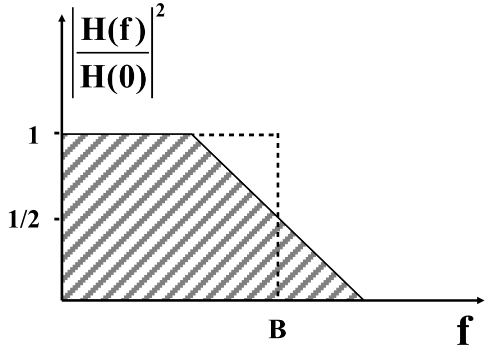
Variance

$$h^2(t) << P(t)$$

$$\int_0^\infty \eta\frac{P(\tau)}{hf}h^2(t-\tau)\partial\tau \approx \frac{\eta}{hf}P(t)\underbrace{\int_0^\infty h^2(t-\tau)\partial\tau}_{2q^2B} = 2qB\underbrace{\eta\frac{q}{hf}}_{\Re}P(t) = \boxed{2qB\Re P(t) = \sigma_{PH}^2(t)}$$

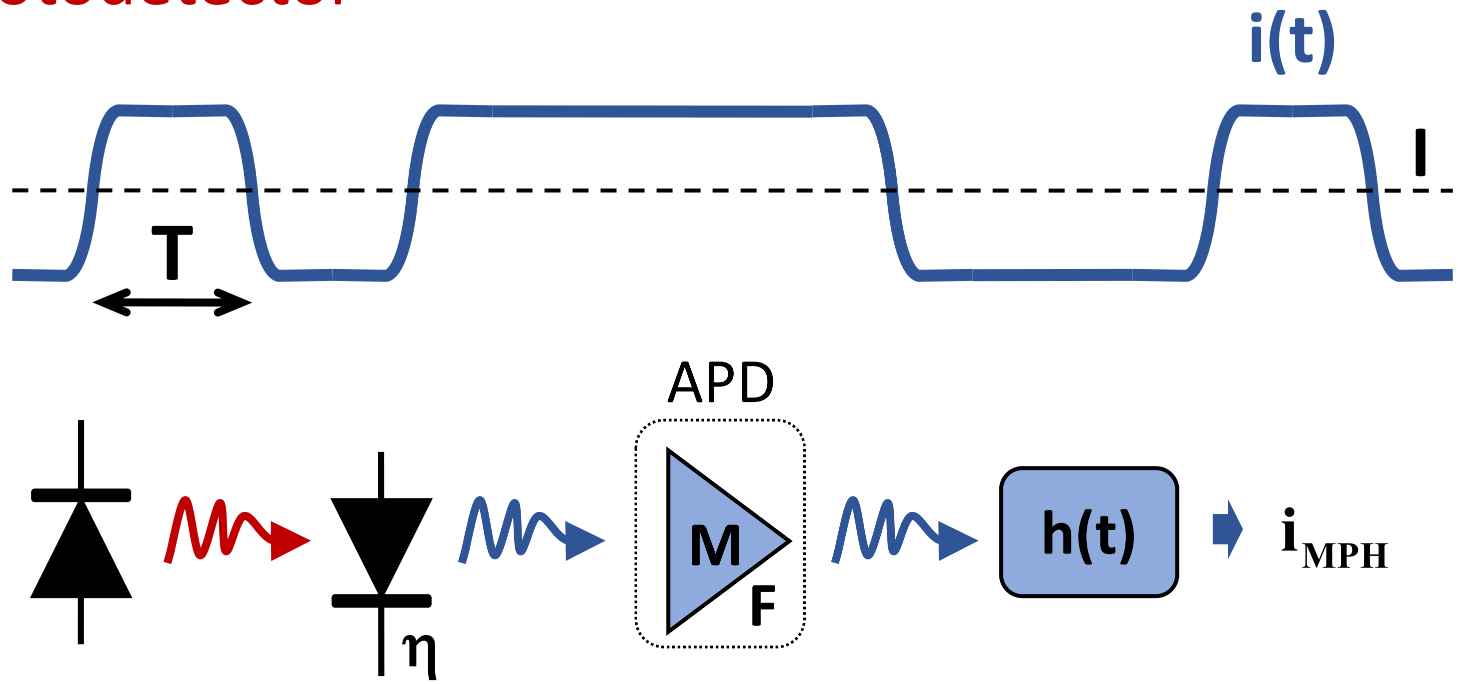
$$B \equiv \frac{1}{2q^2}\int_{-\infty}^{\infty} h^2(t)\partial t = \frac{1}{2}\int_{-\infty}^{\infty}\left|\frac{H(f)}{H(0)}\right|^2\partial f$$

equivalent noise bandwidth



Photodetection Noise

APD photodetector



mean

$$\langle i_{MPH}\rangle(t) = M \cdot \langle i_{PH}\rangle(t)$$

variance

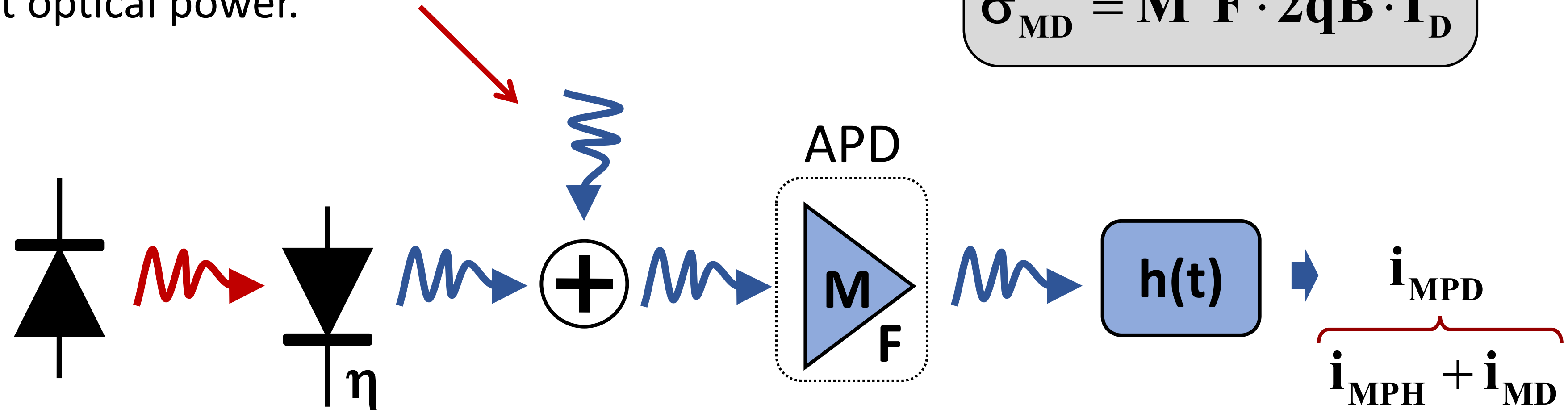
$$\sigma^2_{MPH}(t) = F \cdot M^2 \cdot \sigma^2_{PH}(t)$$

Photodetection Noise

Dark Current

Constant electrical current delivered by the photodetector in the abscence of input optical power.

$$\langle i_{MD} \rangle \equiv M \cdot I_D$$
$$\sigma^2_{MD} = M^2 F \cdot 2qB \cdot I_D$$



mean

$$\langle i_{MPD} \rangle (t) = M \{ \langle i_{PH} \rangle (t) + I_D \}$$

variance

$$\sigma^2_{MPD}(t) = F \cdot M^2 \{ \sigma^2_{PH}(t) + 2qB \cdot I_D \}$$

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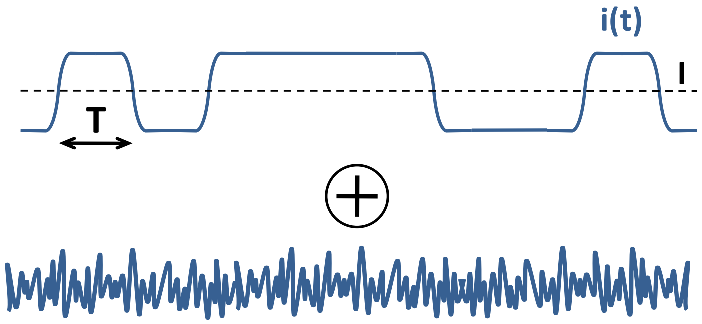
Photodetection Noise

Thermal Noise



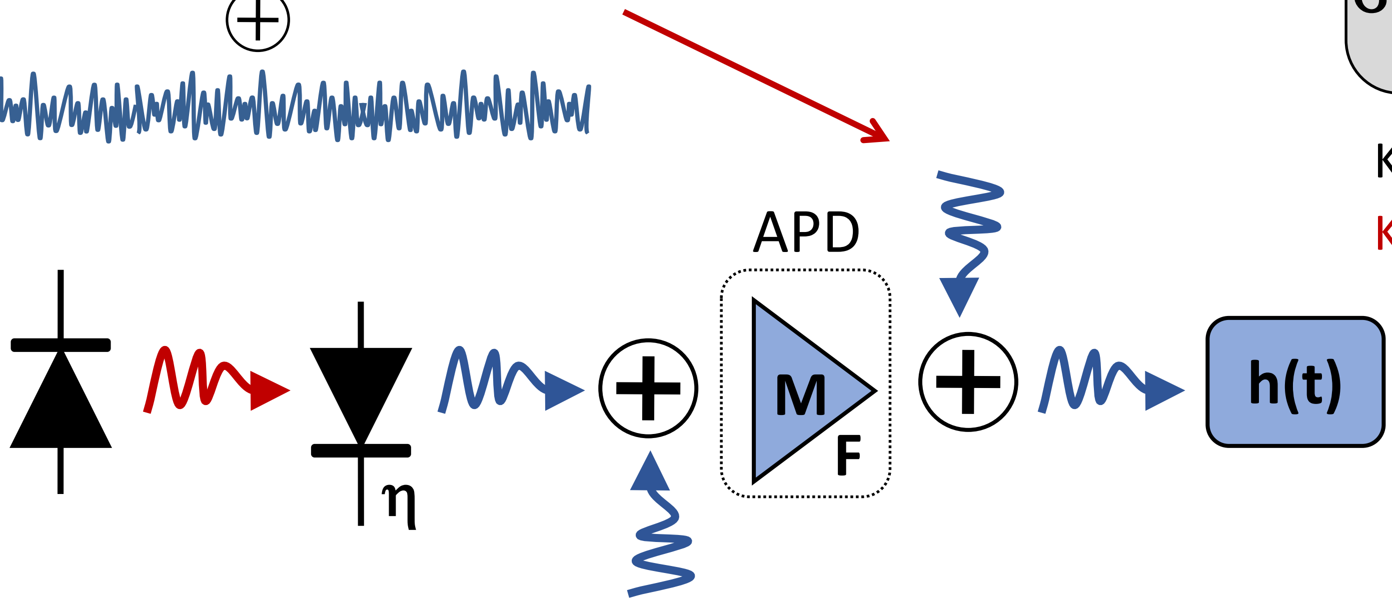
Gaussian Process

$$\langle i_{TH} \rangle (t) = 0$$

$$\sigma^2_{TH}(t) = 4\frac{KT}{R_L}B$$

K = 1.38·10^{-23} J/K

Kelvin Constant



$$i_{APD} = i_{MPD} + i_{TH}$$

mean

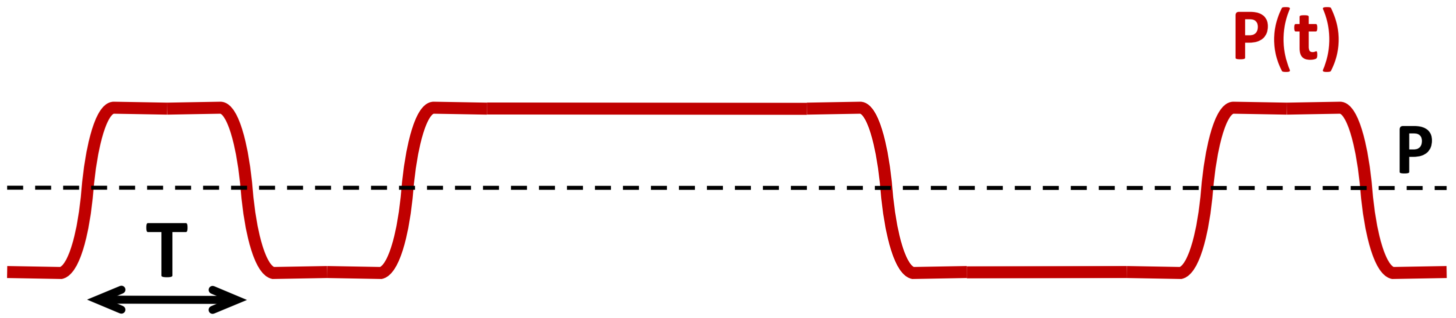
$$\langle i_{APD} \rangle (t) = \langle i_{MPD} \rangle (t)$$

variance

$$\sigma^2_{APD}(t) = \sigma^2_{MPD}(t) + \sigma^2_{TH}(t)$$

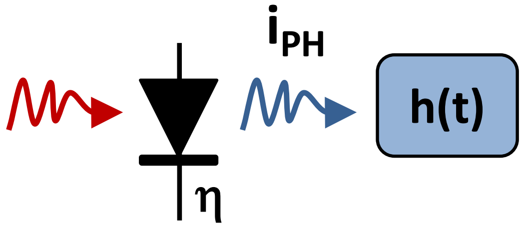
Photodetection Noise

MODULATED SIGNAL



$$P(t) = p(t) * \sum_{k=0}^{\infty} a_k \partial(t - kT_b)$$

PAM

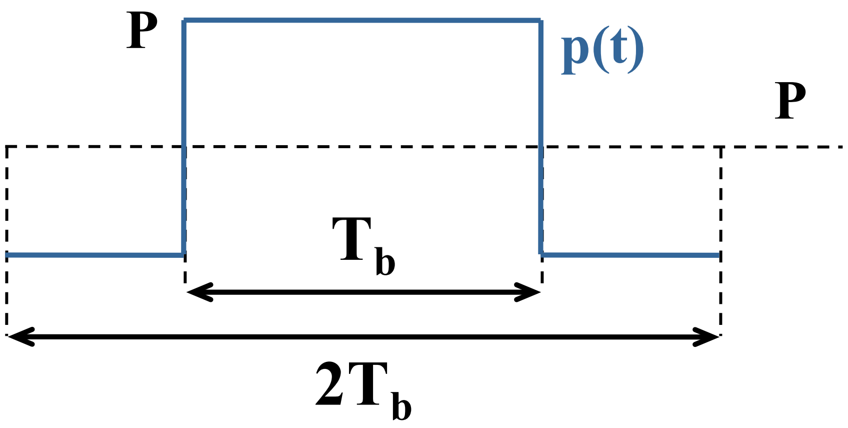


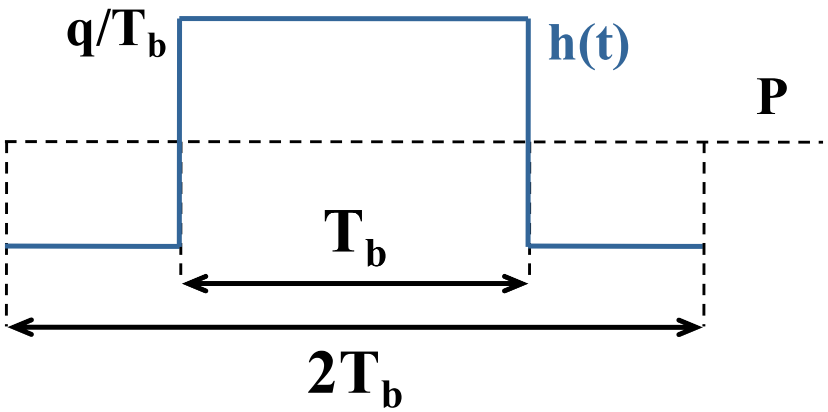
$$\langle i_{PH} \rangle(t) = \frac{\eta}{hf} P(t) * h(t) = \frac{\eta}{hf} \boxed{p(t) * h(t)} * \sum_{k=0}^{\infty} a_k \partial(t - kT_b)$$

$$\sigma_{PH}^2(t) = \frac{\eta}{hf} P(t) * h^2(t) = \frac{\eta}{hf} \boxed{p(t) * h^2(t)} * \sum_{k=0}^{\infty} a_k \partial(t - kT_b)$$

Photodetection Noise

IDEAL PULSES AND FILTERS





Avg. number of photons per bit

$$\langle n_{T_b} \rangle = \frac{1}{hf}\int_0^{\infty} p(\tau)\,\partial\tau = \frac{P}{hf} T_b$$

$$\langle i_{PH} \rangle (t) = \frac{\eta}{hf}\, p(t) * h(t) * \sum_{k=0}^{\infty} \partial(t - kT_b)$$

$$\sigma_{PH}^2(t) = \frac{\eta}{hf}\, p(t) * h^2(t) * \sum_{k=0}^{\infty} \partial(t - kT_b)$$

$$\int_0^{\infty} h(t)\,\partial\tau = q$$

Equivalent noise bandwidth

$$B \equiv \frac{1}{2q^2}\underbrace{\int_0^{\infty} h^2(t)\partial t}_{q^2/T_b} = \frac{1}{2T_b}$$

Photodetection Noise

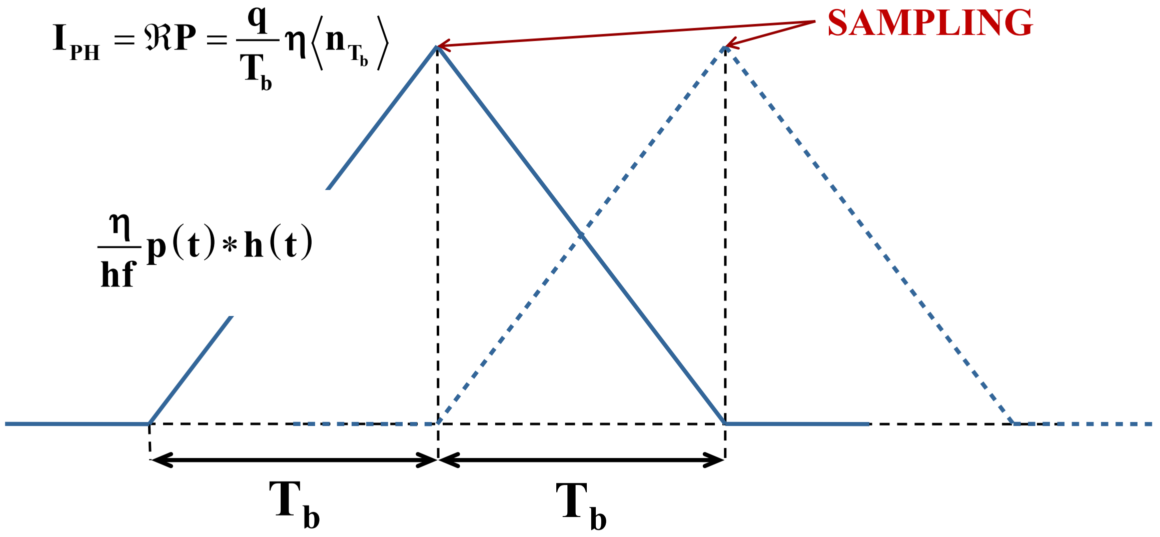
PRIMARY PHOTOCURRENT

Particles – Current Relationship

MEAN

$$\langle i_{PH}\rangle(t)=\frac{\eta}{hf}p(t)*h(t)*\sum_{k=0}^{\infty}\partial(t-kT_b)$$

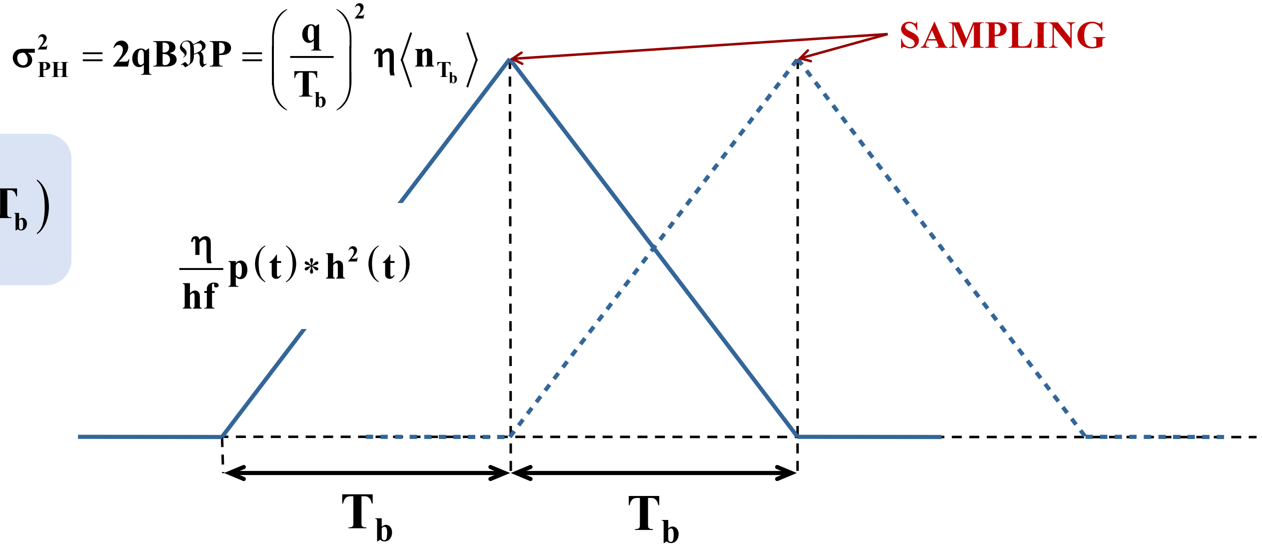
$$\langle i_{PH}\rangle(T_b)=q\underbrace{\frac{\eta}{hf}}_{\Re}P=\frac{q}{T_b}\eta\underbrace{\frac{P}{hf}T_b}_{\langle n_{T_b}\rangle}$$



VARIANCE

$$\sigma_{PH}^2(t)=\frac{\eta}{hf}p(t)*h^2(t)*\sum_{k=0}^{\infty}\partial(t-kT_b)$$

$$\sigma_{PH}^2(T_b)=\underbrace{\frac{q}{T_b}}_{2qB}\,q\underbrace{\frac{\eta}{hf}}_{\Re}P=\left(\frac{q}{T_b}\right)^2\eta\underbrace{\frac{P}{hf}T_b}_{\langle n_{T_b}\rangle}$$



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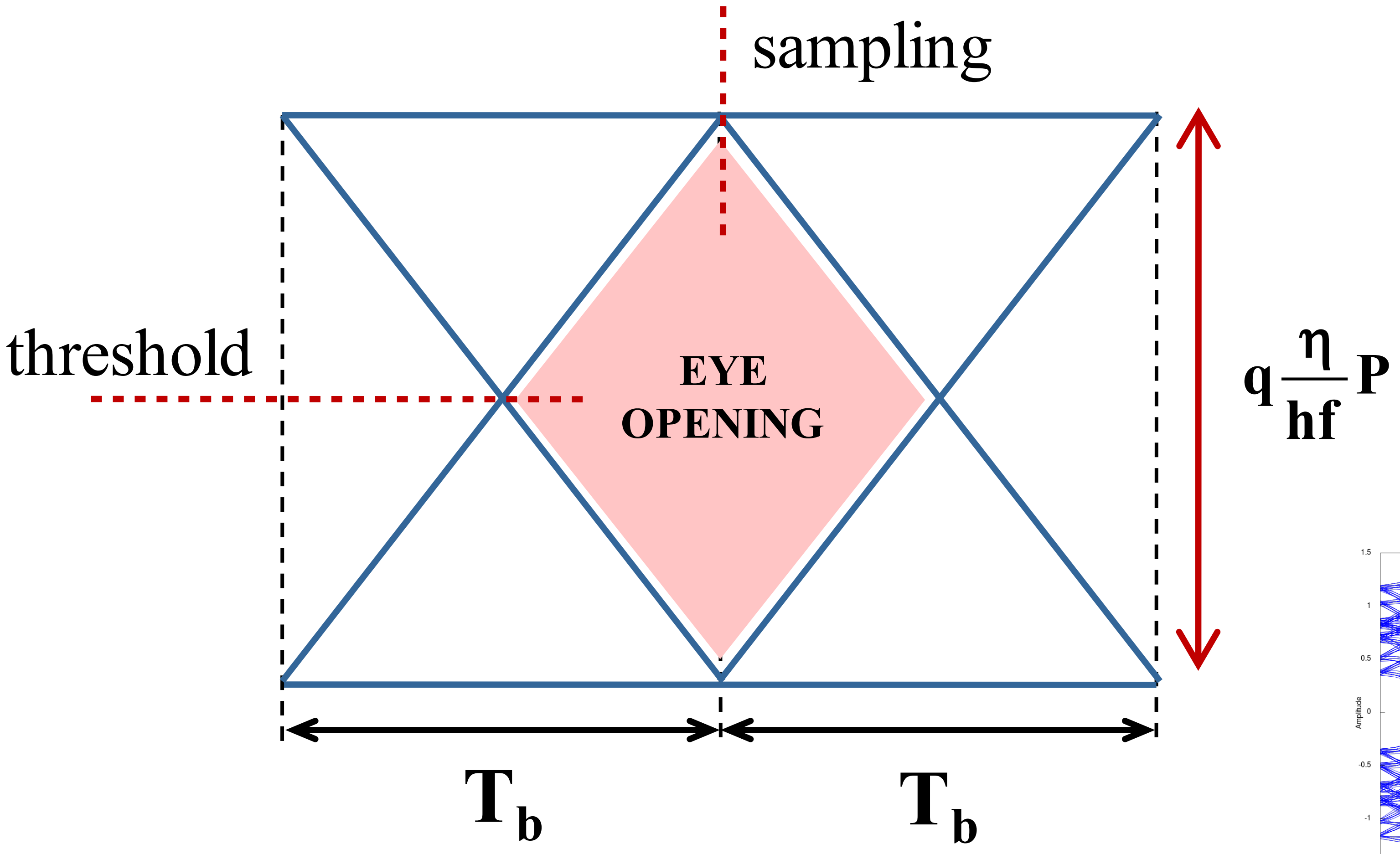
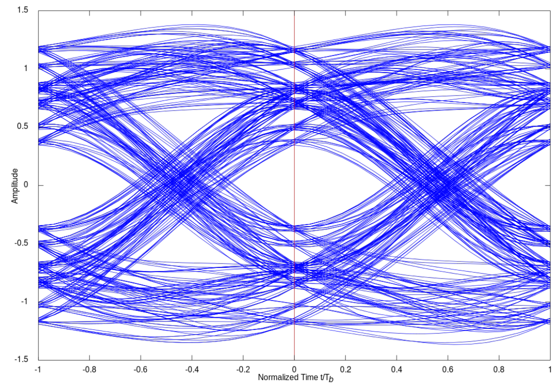
Photodetection Noise

EYE DIAGRAM



multipath interference



https://en.wikipedia.org/wiki/Eye_pattern

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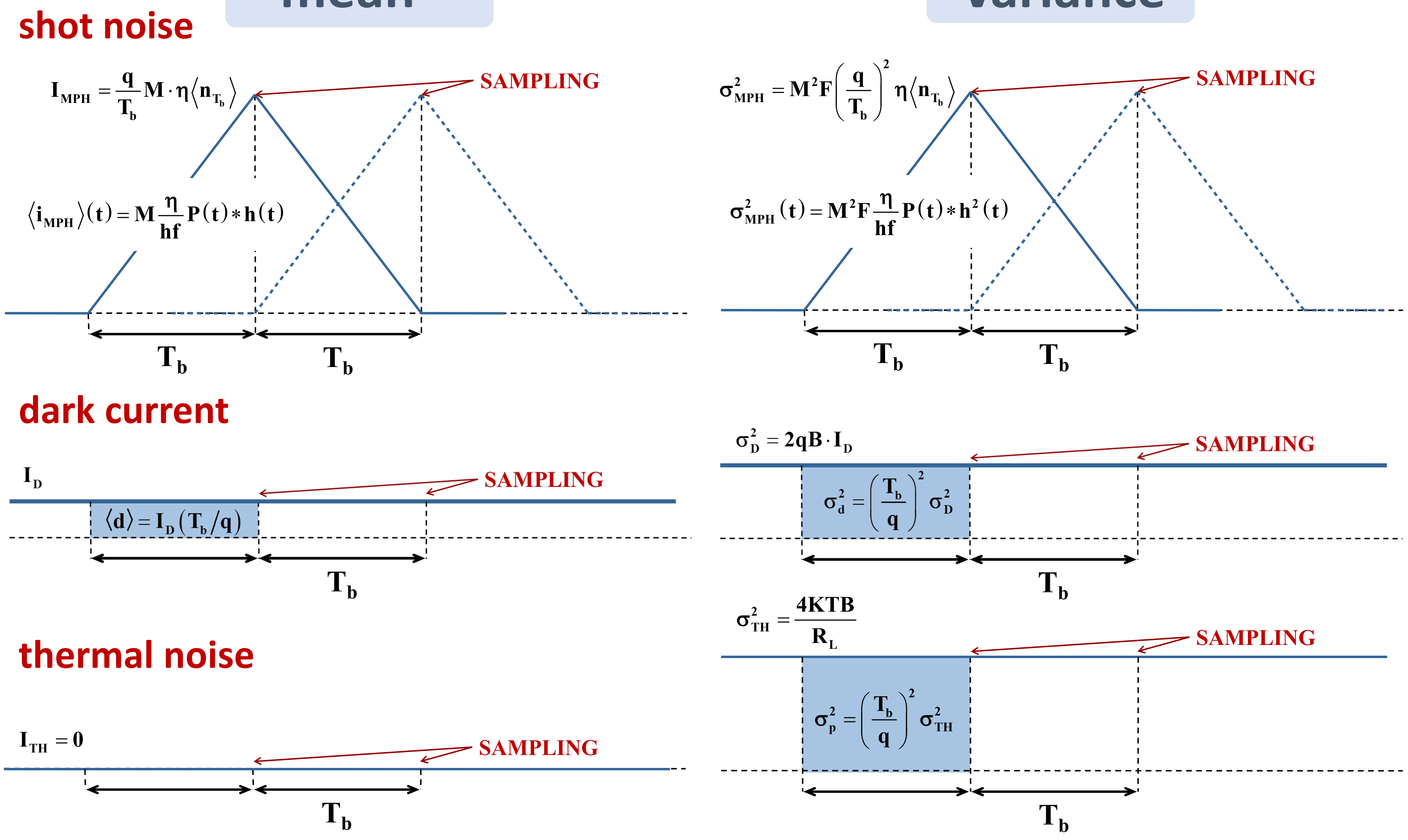
Photodetection Noise

RESULTING CURRENT

mean

variance

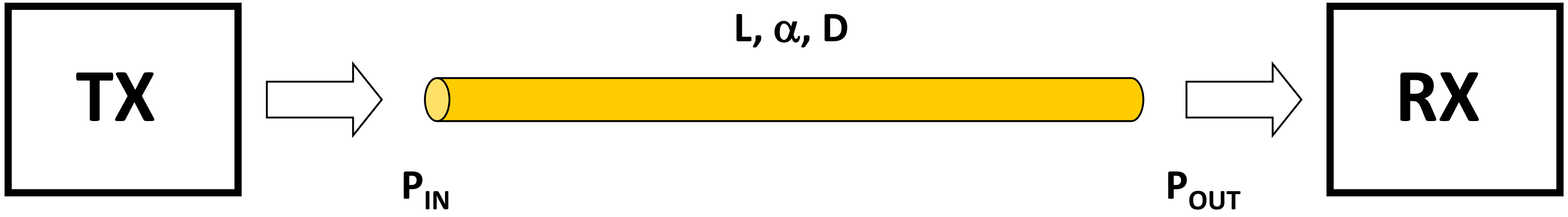


BIT ERROR RATE (BER)

BIT ERROR RATIO (BER)

$$S_i \in \{S_1, S_2, \ldots, S_M\} \qquad M = 2^b$$

b: # bits/symbol



Decision Error Probability

$$p(E) = \sum_{i=1}^{M} p(E/S_i)\, p(S_i)$$

QAM ➡ Gray coding

$$\mathrm{BER} \approx \frac{p(E)}{b}$$

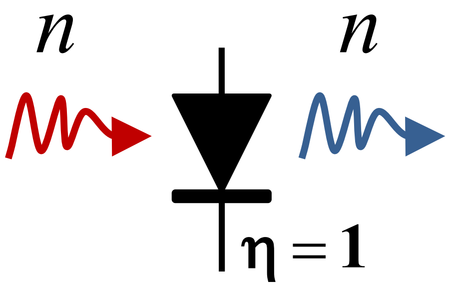
Binary Case (b=1)

$$S_i \in \{0,1\} \;\Rightarrow\; p(E) = p(E/0)\,p(0) + p(E/1)\,p(1) = \mathrm{BER}$$

Quantum Limit

Ideal Receiver (quantum limit)

- Perfectly Coherent Light
- Ideal Intensity Modulation
- Equiprobable Symbols
- 100% Quantum Efficiency
- Ideal Noise Factor (F=1)
- No Dark Current
- No Thermal Noise



Poisson ➡ $p(n=k) = \frac{\langle n \rangle^k}{k!} e^{-\langle n \rangle} \xrightarrow{k=0} e^{-\langle n \rangle}$

$$E(n) = \langle n \rangle$$

decision criterion

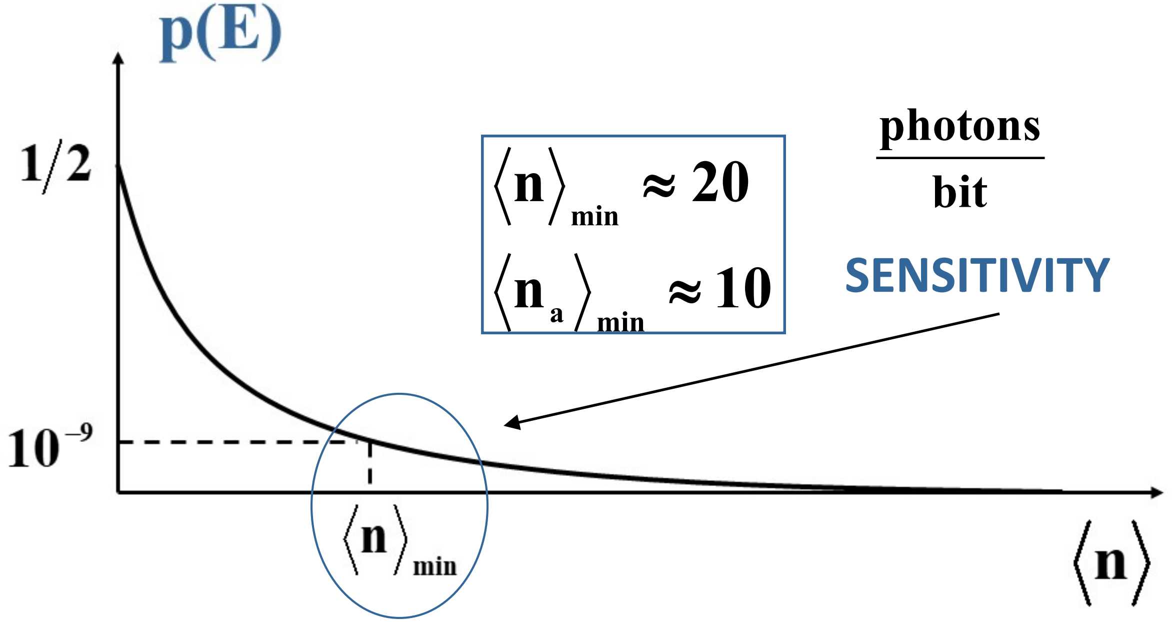
$\mathrm{n} \geq 0 \rightarrow$ "1"

$\mathrm{n} = 0 \rightarrow$ "0"

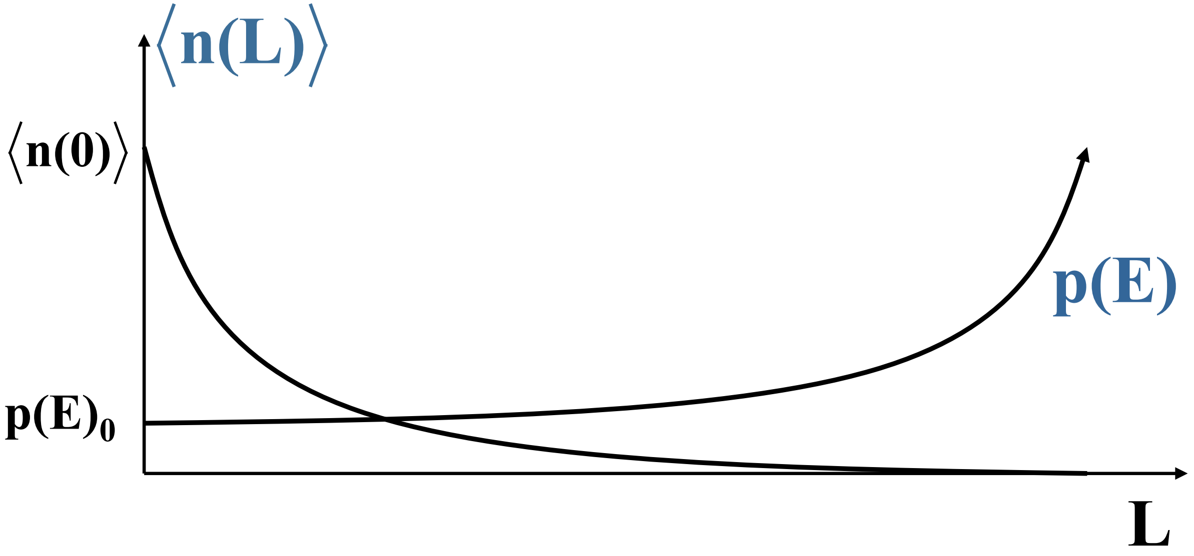
$$\mathrm{p(E)} = \cancel{\mathrm{p(E/0)p(0)}} + \mathrm{p(E/1)p(1)} = \frac{1}{2}\mathrm{e}^{-\langle \mathrm{n} \rangle} = \frac{1}{2}\mathrm{e}^{-2\langle \mathrm{n_a} \rangle}$$

$\langle n_a \rangle$: average number of photons per bit @ RX

Quantum Limit



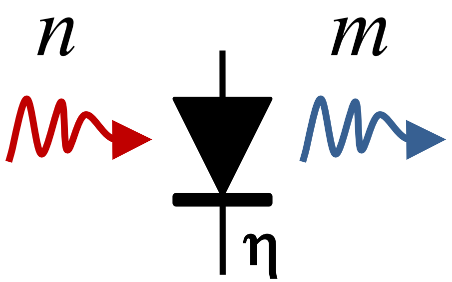
$$P(E)=\frac{1}{2}e^{-\langle n\rangle}=\frac{1}{2}e^{-2\langle n_a\rangle}$$



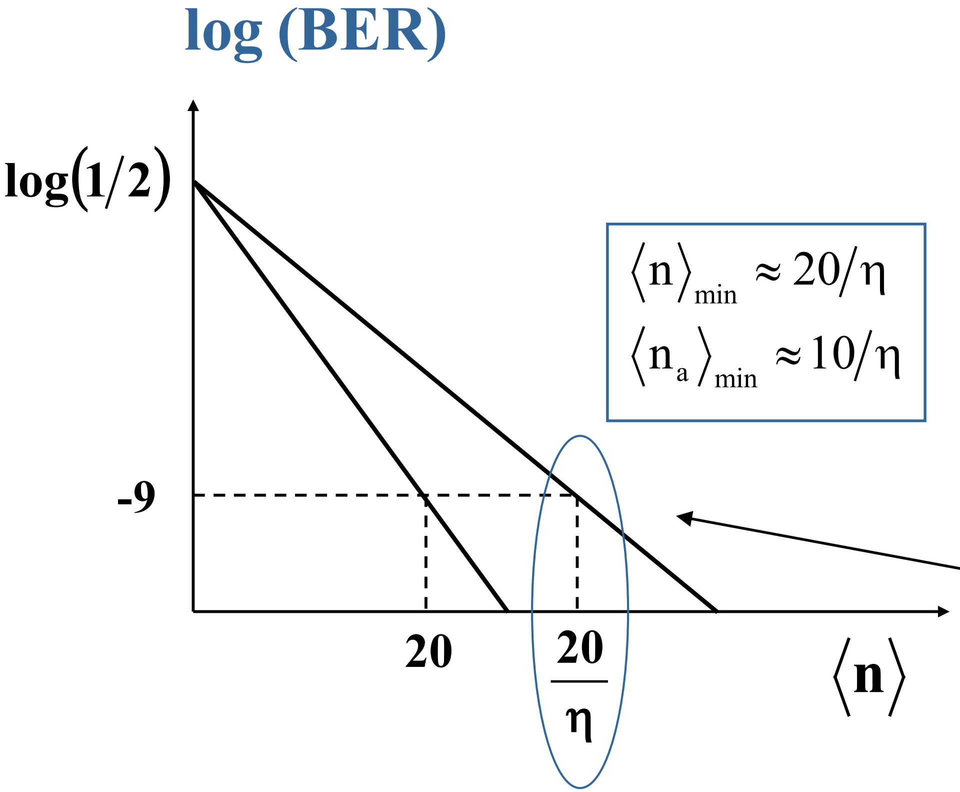
$$\langle n(L)\rangle\equiv\underbrace{\frac{P_0T}{hf}}_{\langle n(0)\rangle}10^{-\alpha L/10}$$

Quantum Limit

Non-Ideal Quantum Efficiency



$$\langle m \rangle = \eta \langle n \rangle$$



$$\langle n \rangle_{min} \approx 20/\eta$$
$$\langle n_a \rangle_{min} \approx 10/\eta$$

$$p(E) = \frac{1}{2} e^{-\langle m \rangle} = \frac{1}{2} e^{-2\langle m_a \rangle}$$
$$p(E) = \frac{1}{2} e^{-\eta\langle n \rangle} = \frac{1}{2} e^{-2\eta\langle n_a \rangle}$$

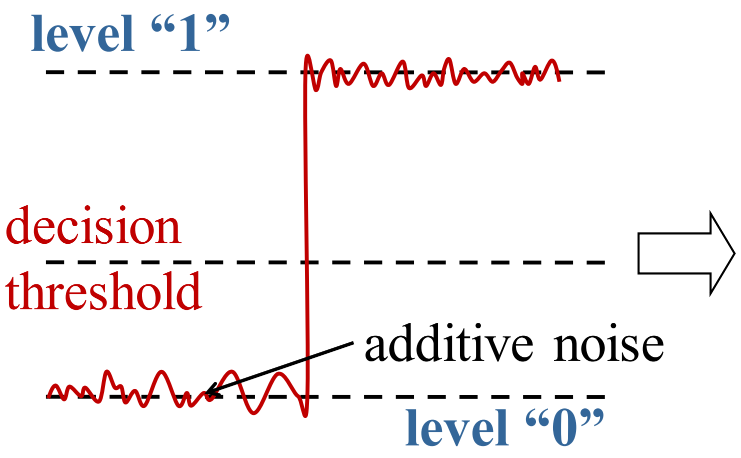
SENSITIVITY

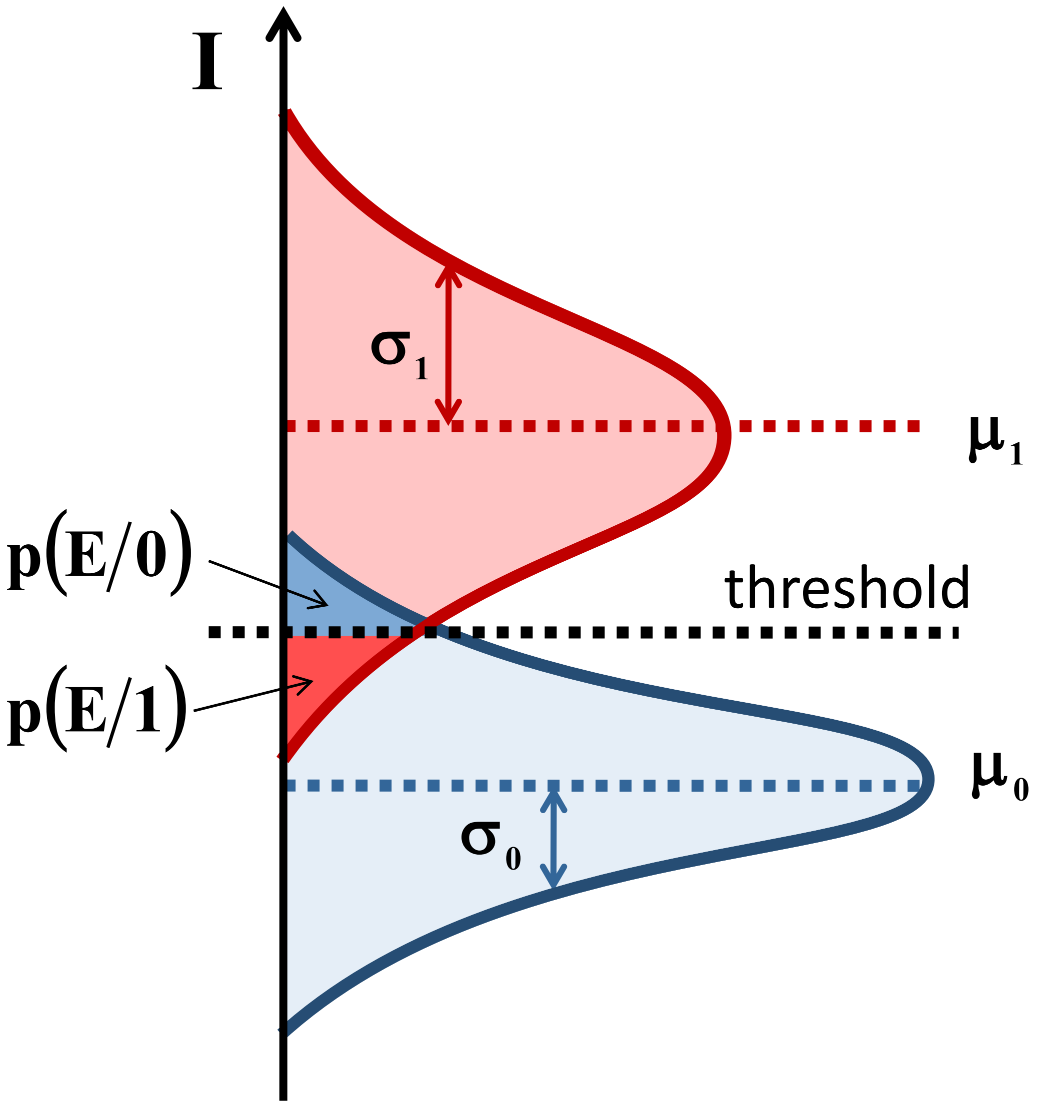
$$\frac{\text{photons}}{\text{bit}}$$

Realistic Receiver

Realistic Receiver

Threshold-based Decision





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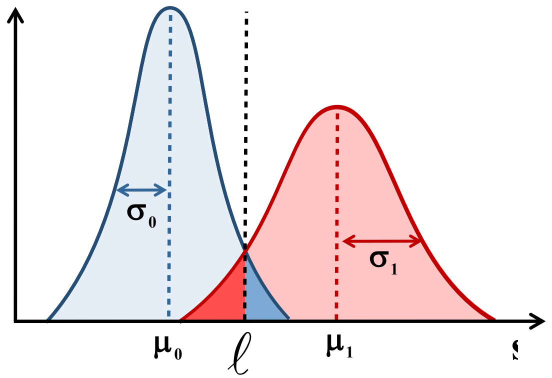
Gaussian Statistics (Normal Distribution)

Realistic Receiver



$$f_0(s) = \frac{1}{\sqrt{2\pi}\sigma_0}\exp\left[-\frac{1}{2}\left(\frac{s-\mu_0}{\sigma_0}\right)^2\right]$$

$$f_1(s) = \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left[-\frac{1}{2}\left(\frac{s-\mu_1}{\sigma_1}\right)^2\right]$$

$$\begin{cases} p(E/0) \equiv \int_{\ell}^{\infty} f_0(s)\partial s \\ p(E/1) \equiv \int_{-\infty}^{\ell} f_1(s)\partial s \end{cases}$$

$$p(E) = p(E/0)p(0) + p(E/1)p(1) = \frac{1}{2}p(E/0) + \frac{1}{2}p(E/1)$$

$$p(E/0)\,p(0) = p(E/1)\,p(1)$$

$$\downarrow \; p(0) = p(1)$$

$$p(E/0) = p(E/1)$$

$$\rightarrow \quad \ell_{OPT} = \frac{\sigma_1\mu_0 + \sigma_0\mu_1}{\sigma_1 + \sigma_0} \xrightarrow{\mu_0=0} \ell_{OPT} = \frac{\mu_1}{1+\sigma_1/\sigma_0} < \frac{\mu_1}{2}$$

$$\sigma_1 \gg \sigma_0 \longrightarrow \ell_{OPT} \approx \mu_0 + \mu_1\frac{\sigma_0}{\sigma_1}$$ (Dominant shot)

$$\sigma_1 \approx \sigma_0 \longrightarrow \ell_{OPT} \approx (\mu_0 + \mu_1)/2$$ (Dominant thermal)

Realistic Receiver

Error Function

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \exp\left[-t^2\right] \partial t$$

$$\mathrm{erf}(x) + \mathrm{erfc}(x) = 1$$

$$\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left[-t^2\right] \partial t$$

$$t \equiv \frac{s-\mu_0}{\sqrt{2}\sigma_0} \qquad \partial s = \sqrt{2}\sigma_0 \partial t$$

$$p(E/0) = \frac{1}{\sqrt{2\pi}\sigma_0} \int_\ell^\infty \exp\left[-\frac{1}{2}\left(\frac{s-\mu_0}{\sigma_0}\right)^2\right] \partial s = \frac{1}{\sqrt{\pi}} \int_{\ell_T}^\infty \exp\left[-t^2\right] \partial t = \frac{1}{2} \mathrm{erfc}(\ell_T)$$

$$\ell_T = \frac{\ell_{OPT} - \mu_0}{\sqrt{2}\sigma_0} = \frac{\dfrac{\sigma_1\mu_0 + \sigma_0\mu_1}{\sigma_1 + \sigma_0} - \mu_0}{\sqrt{2}\sigma_0} = \frac{\sigma_1\mu_0 + \sigma_0\mu_1 - \mu_0\sigma_1 - \mu_0\sigma_0}{(\sigma_1 + \sigma_0)\sqrt{2}\sigma_0}$$

$$= \frac{\sigma_0\mu_1 - \mu_0\sigma_0}{(\sigma_1 + \sigma_0)\sqrt{2}\sigma_0} = \frac{1}{\sqrt{2}} \boxed{\frac{\mu_1 - \mu_0}{\sigma_1 + \sigma_0}} \; Q$$

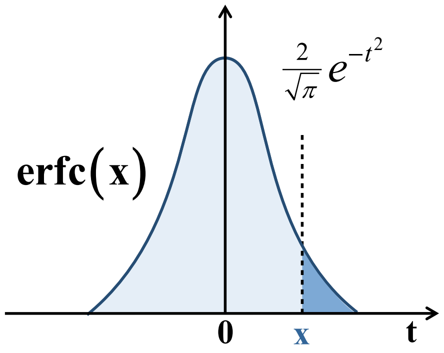
$$p(E/1) = p(E/0) \Rightarrow p(E) = p(E/0) = \frac{1}{2}\mathrm{erfc}(\ell_T) = \frac{1}{2}\mathrm{erfc}\left(\frac{Q}{\sqrt{2}}\right)$$

Realistic Receiver

Quality Parameter (Q)

$$Q \equiv \frac{\mu_1 - \mu_0}{\sigma_1 + \sigma_0}$$

$$BER \equiv \frac{1}{2} \mathrm{erfc}\left(\frac{Q}{\sqrt{2}}\right)$$

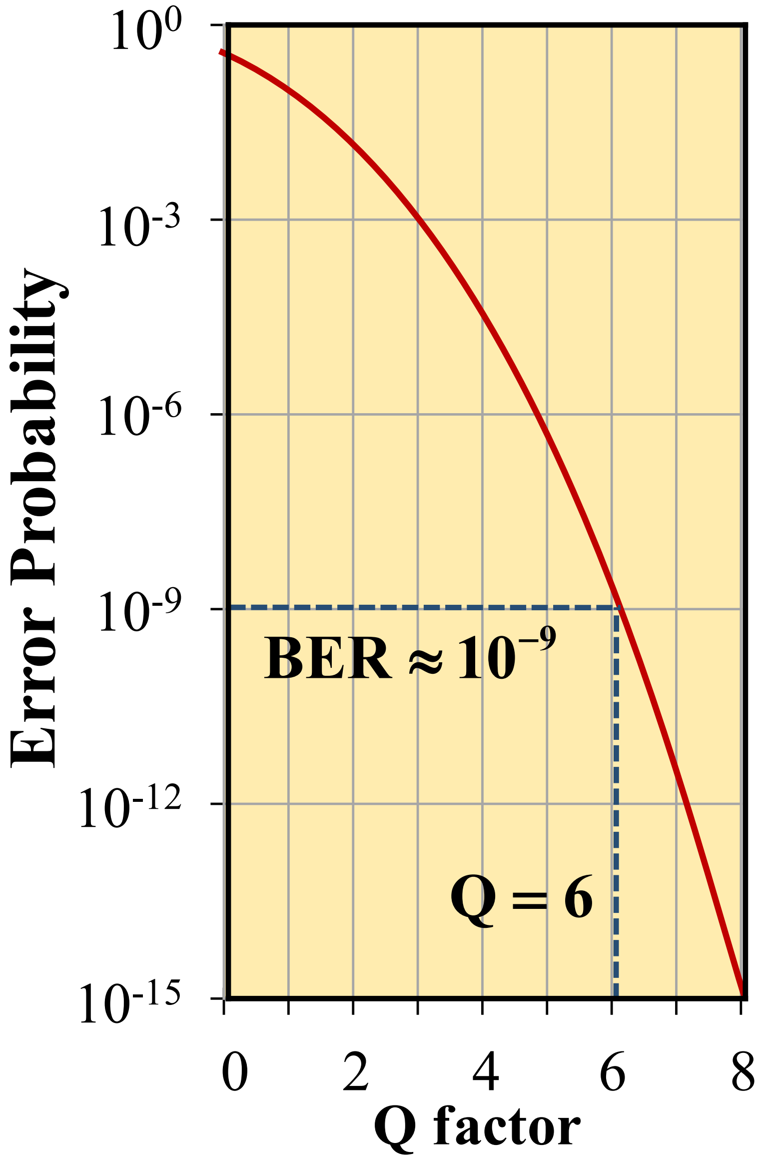


$$Q = 0 \rightarrow BER = \frac{1}{2}$$

$$Q = \infty \rightarrow BER = 0$$

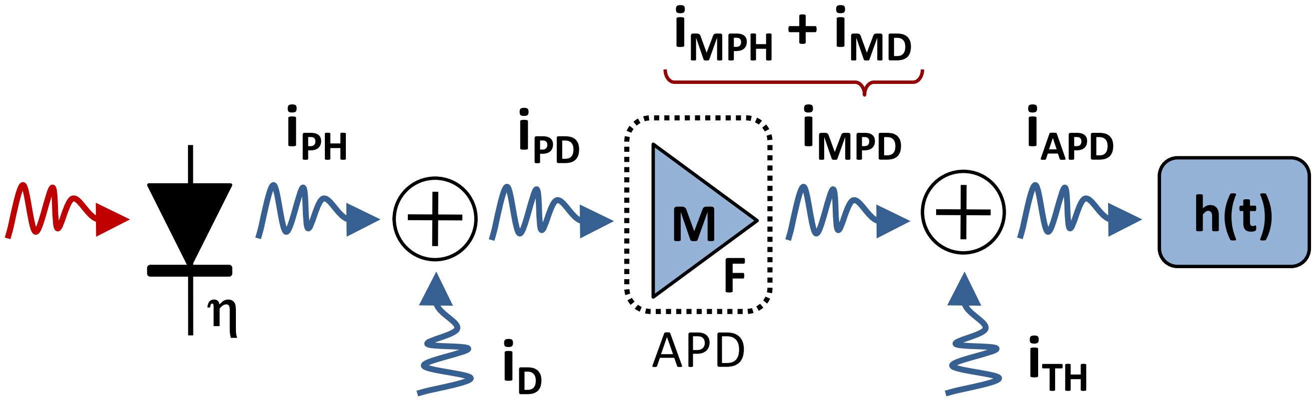
$$erfc(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left[-t^2\right] \partial t$$

Complementary Error Function



Realistic Receiver

Complete Receiver Model



$$\langle i_{TH} \rangle = 0$$

$$\sigma_{TH}^2 = \langle i_{TH}^2 \rangle = 4\frac{KT}{R_L}B$$

B: equivalent noise bandwidth

Thermal Noise

(Additive White Gaussian Noise AWGN)

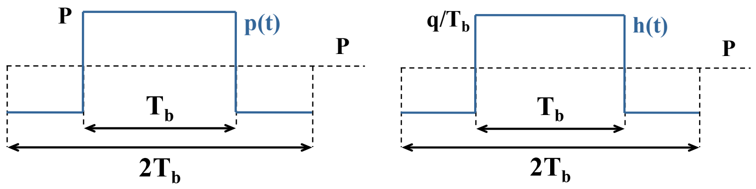
Shot Noise vs. Thermal Noise

- Both depend on receiver’s bandwidth
- Both show a uniform spectrum in the whole frequency band
- Thermal noise is independent of optical power while shot noise is proportional to it

Realistic Receiver

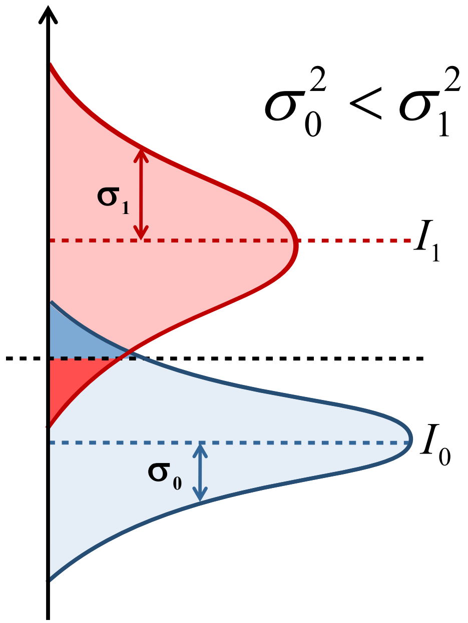
mean $I = M\left(\Re P + I_D\right)$

variance $\sigma^2 = F \cdot M^2\left(2qB\Re P + I_D\right) + \sigma_{th}^2$



$$\text{"0"}\begin{cases} I_0 = M \cdot I_D \\ \sigma_0^2 = F \cdot M^2 I_D + \sigma_{th}^2 \end{cases} \sim \textbf{Gauss}$$

$$\text{"1"}\begin{cases} I_1 = M\left(\Re P_1 + I_D\right) \\ \sigma_1^2 = F \cdot M^2 2qB\left(\Re P_1 + I_D\right) + \sigma_{th}^2 \end{cases} \sim \textbf{Gauss}$$



$$Q \equiv \frac{I_1 - I_0}{\sigma_1 + \sigma_0} = \frac{M \cdot \Re P_1}{\sqrt{F \cdot M^2\left(2qB\Re P_1 + I_D\right) + \sigma_{th}^2} + \sigma_{th}}$$

$$\langle n_1 \rangle = \frac{P_1}{hf} T_b$$

$$B = \frac{1}{2T_b}$$

Realistic Receiver

Receiver Sensitivity – PIN

$$I_D = 0$$

$$I_1 = \Re P_1 \qquad \sigma_1 = \sqrt{2qB\Re P_1 + \sigma_{th}^2}$$

$$I_0 = 0 \qquad \sigma_0 = \sigma_{th}$$

$$Q_{PIN} \equiv \frac{I_1 - I_0}{\sigma_1 + \sigma_0} = \frac{\Re P_1}{\sqrt{2qB\Re P_1 + \sigma_{th}^2} + \sigma_{th}} \geq Q \Rightarrow P_1 \geq \frac{2Q}{\Re}\left(qBQ + \sigma_{th}\right)$$

Average Power

$$P_a \equiv \frac{P_1 + P_0}{2} = \frac{P_1}{2} \Rightarrow \boxed{P_a \geq \frac{Q}{\Re}\left(qBQ + \sigma_{th}\right)}$$

$$\langle n_1 \rangle = P_1 \frac{T_b}{hf} \qquad B = \frac{1}{2T_b} \qquad \Re = \eta \frac{q}{hf}$$

Photons per bit

$$\Rightarrow \langle n_1 \rangle \geq \frac{2Q}{\eta}\left(\frac{Q}{2} + \sigma_{th}\frac{T_b}{q}\right)$$

$$BER = 10^{-9} \; (Q = 6) \Rightarrow \langle n_1 \rangle \geq \frac{12}{\eta}\left(3 + \sigma_{th}\frac{T_b}{q}\right) \xrightarrow[\eta = 1]{\sigma_{th} = 0} \langle n_1 \rangle \geq 36$$

Wrong model

Receiver Sensitivity – APD

Realistic Receiver

$$I_D = 0$$

$$I_1 = M \cdot \Re P_1 \qquad \sigma_1 = \sqrt{M^2 F \cdot 2qB\Re P_1 + \sigma_{th}^2}$$

$$I_0 = 0 \qquad \sigma_0 = \sigma_{th}$$

$$P_a \equiv \frac{P_1 + P_0}{2} = \frac{P_1}{2}$$

Average Power

$$Q_{APD} \equiv \frac{I_1 - I_0}{\sigma_1 + \sigma_0} = \frac{M \cdot \Re P_1}{\sqrt{M^2 F \cdot 2qB\Re P_1 + \sigma_{th}^2} + \sigma_{th}} \geq Q \Rightarrow P_a \geq \frac{Q}{\Re}\left(F \cdot qBQ + \frac{\sigma_{th}}{M}\right)$$

$$\langle n_1 \rangle = P_1 \frac{T_b}{hf} \qquad B = \frac{1}{2T_b} \qquad \Re = \eta \frac{q}{hf}$$

Photons per bit

$$\Rightarrow \langle n_1 \rangle \geq \frac{2Q}{\eta}\left(F\frac{Q}{2} + \frac{\sigma_{th}}{M}\frac{T_b}{q}\right)$$

Gain Optimization

$$F(M) \Rightarrow P_a \geq \frac{Q}{\Re}\left(F(M) \cdot qBQ + \frac{\sigma_{th}}{M}\right) \xrightarrow{\frac{\partial}{\partial \mathbf{M}} = 0} \mathbf{M}_{\mathbf{opt,S}} \quad \mathbf{S}_{\mathbf{APD,min}}$$

$$F = M^x$$

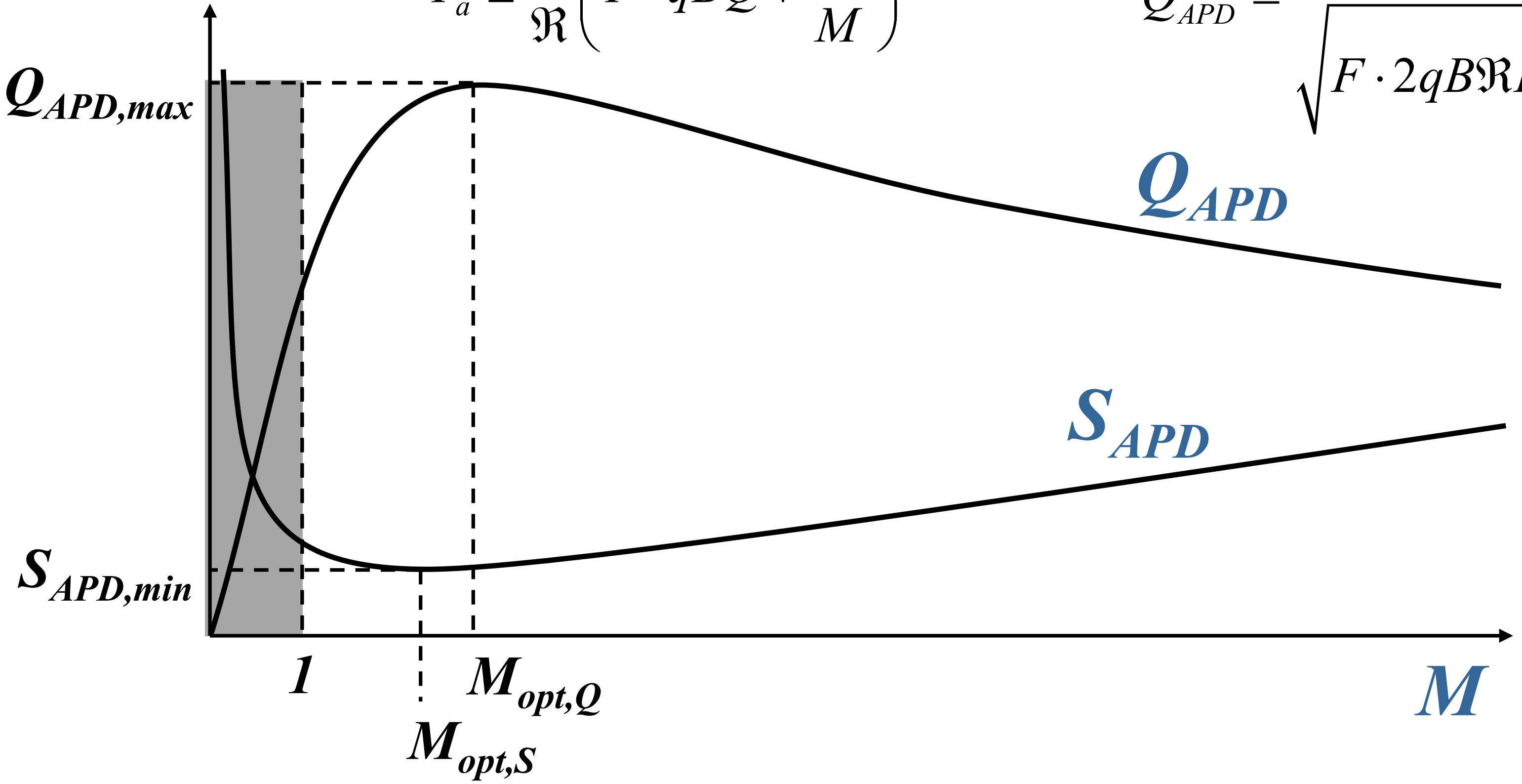
Realistic Receiver

Gain Optimization Summary

Optimum Gain depends on the criterion: Sensitivity or BER

$$P_a \geq \frac{Q}{\Re}\left(F \cdot qBQ + \frac{\sigma_{th}}{M} \right)$$

$$Q_{APD} = \frac{\Re P_1}{\sqrt{F \cdot 2qB\Re P_1 + \frac{\sigma_{th}^2}{M^2}} + \frac{\sigma_{th}}{M}}$$



APD gives better performance than PIN as long as $M_{opt} > 1$